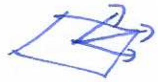
<u>Defn</u> Vectors $v_1, v_2, \dots, v_n$ in a vector space V are <u>linearly independent</u> if
$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ implies all $a_i = 0$.

<u>Intuition</u>: linear independence $\Leftrightarrow$ vectors point in different directions.

<u>Example</u> $\mathbb{R}^3$:  indep.  not indep.

<u>Defn</u> a set of vectors $v_1, v_2, \dots, v_n$ <u>span</u> V if any vector $v \in V$ can be written as
$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ for scalars a_i.

<u>Thm/Fact</u> A basis for V is a set of vectors $(v_1, v_2, \dots v_n)$ which are both linearly independant <u>and</u> span V.

<u>Fact</u> every linear map $T: V \to W$ can be written as a matrix.

choose bases: $v_1, \dots v_n$ for V and $w_1, \dots w_m$ for W

so any vector $v \in V$ can be written as $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, think of this as a column vector $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Any vector $w \in W$ can be written as $w = a_1w_1 + a_2w_2 + \cdots + a_mw_m$
think of this as a column vector $\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$.

T maps each basis vector $v_i \in V$ to $T(v_i) \in W$.

as the $w_1 \dots w_m$ span, each $T(v_i) = a_{1i}w_1 + a_{2i}w_2 + \cdots + a_{mi}w_m \Leftrightarrow \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix}$

<u>claim</u> T corresponds to $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}$.

<u>check</u> $v \in V$, then $v = x_1v_1 + x_2v_2 + \cdots + x_nv_n \Leftrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x$.

then $T(v) = T(x_1v_1 + \cdots + x_nv_n) = x_1T(v_1) + \cdots + x_nT(v_n)$.

$$= x_1(a_{11}w_1 + \cdots + a_{m1}w_m) + \cdots + x_n(a_{1n}w_1 + \cdots + a_{mn}w_m)$$

in column vector notation.

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax,$$

as required. $\square$.

warning the matrix A for T depends on the choice of basis for V and W.

Q: what if we choose different bases?

key fact: change of basis is a linear map!

let V have basis $E = (e_1, e_2, \dots e_n)$ and ~~$E = (\epsilon_1, \epsilon_2, \dots \epsilon_n)$~~ $F = (f_1, f_2, \dots f_n)$

then $v \in V$ can be written as $v = a_1 e_1 + a_2 e_2 + \dots + a_n e_n \leftrightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_E$

or $v = b_1 f_1 + b_2 f_2 + \dots + b_n f_n \leftrightarrow \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_F$.

but each vector e_i lies in V, so each $e_i = c_{1i} f_1 + c_{2i} f_2 + \dots + c_{ni} f_n$.

so $v = a_1(c_{11} f_1 + \dots + c_{n1} f_n) + \dots + a_n (c_{n1} f_1 + \dots + c_{nn} f_n)$.

so

$$= \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \\ c_{n1} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_E = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_F.$$

columns of this matrix are basis vectors of E written in F.

so if $T: V \to W$ and choose bases E, F for V and G, H for W, get

$$\begin{tikzcd} V_E \arrow[r, "A"] \arrow[d, "C"'] & W_G \arrow[d, "D"] \\ V_F \arrow[r, "B"'] & W_H \end{tikzcd}$$

so $B = DAC^{-1}$

fact: change of bases map has an inverse, which is a linear map.

warning: big difference between $T: V \to W$
and $T: V \to V$ ← usually want same basis for each V.

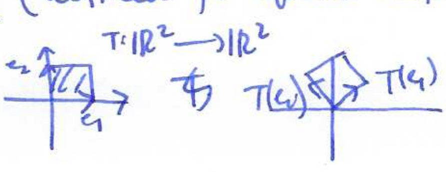
so get

$$\begin{tikzcd} V_E \arrow[r, "A"] \arrow[d, "C"'] & V_E \arrow[d, "C"] \\ V_F \arrow[r, "B"'] & V_F \end{tikzcd}$$

so $B = CAC^{-1}$.

Determinants (defined for square matrices, in particular for change of basis matrices)

Intuition



the unit square/cube in $\mathbb{R}^2$ has image under T which is a parallelepiped.

Fact $\det(A)$ = volume of image of unit cube (signed, sign depends on preserving / reversing orientation).

Computing the determinant (recursive method – slow).

1×1 case: $\det([a_{11}]) = a_{11}$

A $n\times n$: $\det(A) = \sum_{k=1}^{n} (-1)^{k+1} a_{1k} \det(A_{1k})$ $1k$ minor. i.e. delete 1st row, kth column.

Examples $\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad-bc$ $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$.

Useful properties (can use this as defn).

Defn the determinant is the unique real valued function $\det : M_{n\times n} \to \mathbb{R}$ s.t.

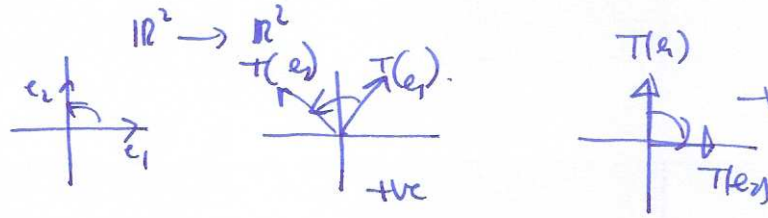
c_i columns of A.

1) $\det[c_1 \dots \lambda c_i \dots c_n] = \lambda \det[c_1\, c_2 \dots c_n]$ ← multiply a col by λ multiplies det by λ.

2) $\det[c_1 \dots c_k + \lambda c_i \dots c_n] = \det[c_1 \dots c_n]$ for $i \neq k$ ← add a col to another one doesn't change det.

3) $\det(I) = 1$.

Orientation (example).



Useful property • $\det(AB) = \det(A)\det(B)$.

Useful property (for $n\times n$ matrices) $\det(A) \neq 0 \iff A$ is invertible.

Thm A $n\times n$ matrix. The following are equivalent:

1) A has an inverse.
2) $\det(A) \neq 0$
3) $\ker(A) = 0$
4) if $b \in \mathbb{R}^n$ then there is a unique $x \in \mathbb{R}^n$ s.t. $Ax = b$.
5) the columns of A are linearly independent.
6) the rows of A are linearly independent.
7) A^T is invertible (transpose)
8) all of the eigenvalues of A are non-zero (to be defined!)

if $T:V\to V$ then
Note value of $\det(A)$ depends on basis, but condition $\det(A)\neq 0$ does not.

Eigenvalues and eigenvectors

Example $A\begin{bmatrix} -1 & -2 & -2 \\ 12 & 7 & 4 \\ -9 & -3 & 0 \end{bmatrix}$ corresponds to $T: V\to V$.
can choose different basis s.t. T corresponds to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ B.

$T: V_E \to V_E$. $\qquad T: V_F\to V_F$.

Defn Let $T:V\to V$ be a linear map. $v \overset{\neq 0}{\text{is}}$ an eigenvector with eigenvalue λ if $T(v)=\lambda v$. **warning** $v=0$ is never an eigenvector. $\lambda=0$ ok though.

Example $\begin{bmatrix} -2 & -2 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Q: how do we find eigenvectors / values? **Note** if $Ax=\lambda x$ $(x\neq 0)$ then $Ax=\lambda I x$.

so $Ax-\lambda I x = 0$

so $(A-\lambda I)x=0$

i.e. $\ker(A-\lambda I)\neq 0$

i.e $\det(A-\lambda I)=0$.

Prop^n λ is an eigenvalue of A iff. λ is a root of $P(t)=\det(\underset{A-tI}{\cancel{tI-A}}) \leftarrow$ called characteristic polynomial of A.

Thm If A and B differ by a change of basis then they have the same characteristic polynomials (i.e. same eigenvalues).

Proof suppose $A=C^{-1}BC$. Then $\det(A-tI)=\det(C^{-1}BC-tI)$.

$$
\begin{aligned}
&= \det(C^{-1}BC - tC^{-1}C). \\
&= \det(C^{-1}(B-tI)C). \\
&= \det(C^{-1})\det(B-tI)\det(C) \\
&= \det(C^{-1})\det(C)\det(B-tI). \\
&= \det(C^{-1}C)\det(B-tI) \\
&= \det(B-tI) \quad \square.
\end{aligned}
$$

Warning same eigenvalues $\not\Rightarrow$ differ by a change of basis.

Examples $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.