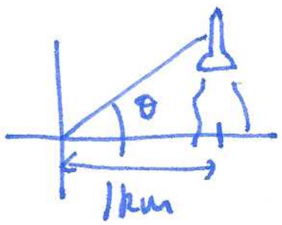
advice ① give things names

② write down relations between them and use implicit differentiation

③ plug in numbers as necessary

Example

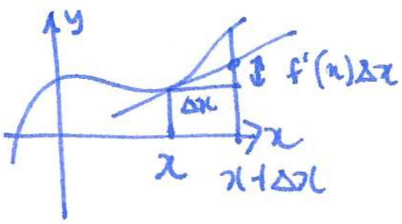


Q: if angle is $\theta = \frac{\pi}{3}$ and rate of change is $\frac{d\theta}{dt} = \frac{1}{2}$ rad/sec

how fast is the rocket going?

$\frac{h}{1} = \tan\theta \qquad \frac{dh}{dt} = \sec^2\theta \, \frac{d\theta}{dt} \qquad \frac{dh}{dt} = \sec^2\left(\frac{\pi}{3}\right)\frac{1}{2} \approx 0.1$ km/sec

§4.1 Linear approximation



if $f(x)$ is differentiable at x, and Δx is small, then

$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$

so change in f is $\Delta f \approx f(x+\Delta x) - f(x)$

$\approx f'(x)\Delta x$

Example estimate $\sqrt{103}$

$f(x) = \sqrt{x} = x^{1/2} \qquad f(100) = 10 \qquad$ so $\Delta f \approx f'(x)\,\Delta x$

$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(100) = \frac{1}{20} \qquad \frac{1}{20}\cdot 3$

so $\sqrt{103} \approx 10 + \frac{3}{20} = 10.15$

Example you make an 18" pizza, if the diameter is accurate to ±0.4in, how much pizza do you gain/lose?

$A = \pi r^2, \quad 2r = D \quad A = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}, \quad A'(D) = \frac{2\pi D}{4} = \frac{\pi D}{2}$

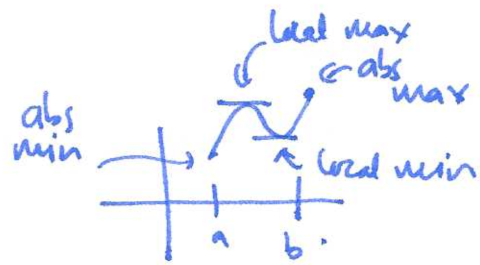
$\Delta A = A'(18)\cdot\Delta D = \frac{1}{2}\pi . 18 . 0.4 \approx 11 \text{ in}^2$

Q: is this good or bad? absolute error = 11

percentage error = $\left|\frac{\text{absolute error}}{\text{actual value}}\right| \times 100 = \frac{11}{\pi . 18^2/4} \times 100 \sim 4\%.$

<u>Observation</u>: when is the linear approximation a good approximation?

$|f''(x)|$ small $\qquad\qquad$ $|f''(x)|$ big



§4.2 Extreme values

suppose $f(x)$ is defined on a closed interval $[a,b]$

<u>Defn</u> $f(c)$ is the <u>absolute max</u> if $f(c) \geq f(x)$ for all $x \in [a,b]$

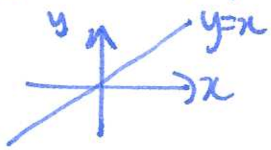
$f(c)$ is the <u>absolute min</u> if $f(c) \leq f(x)$ for all $x \in [a,b]$

<u>note</u>: Q: <u>where</u> is the local max/min? $\leftarrow$ want x value

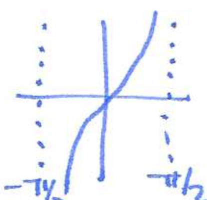
Q: <u>what</u> is the local max/min? $\leftarrow$ want y value

<u>warning</u>: some functions do not have any local max or min

<u>Examples</u> $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$



$f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $x \mapsto \tan x$



<u>Thm</u> If $f(x)$ is continuous on <u>a closed and bounded</u> interval, then $f(x)$ has both an absolute max and an absolute min.

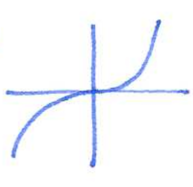
<u>Defn</u> $f(x)$ has a <u>local max</u> at $x=c$ if there is a small interval containing c such that $f(c)$ is an absolute max on this interval.

$f(x)$ is a <u>local min</u> at $x=c$ if there is a small interval containing c such that $f(c)$ is an absolute min on this interval.

<u>Defn</u> we say that $x=c$ is a critical point if $f'(c)=0$ (or undefined)

<u>warning</u> $f'(c)=0 \not\Rightarrow c$ is a local max or min.

<u>Example</u> $y=x^3$



$f'(x)=3x^2$ $\quad$ $f'(0)=0$ $\quad$ but $x=0$ not local max or min

• How to find the absolute max or min of a differentiable function on a closed interval $[a,b]$ ① find critical points, evaluate function at these points. ② check endpoints.

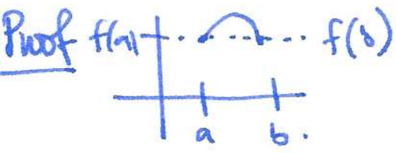
Examples ① find abs max/min of $2x^3 - 15x^2 + 24x + 7$ on $[0,3]$

② $x^2 - 8$ on $[1,4]$

③ $\sin(x)\cos(x)$ on $[0,\pi]$

Thm (Rolle's Thm) suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) \neq f(b)$ then there is a $c \in (a,b)$ s.t. $f'(c) = 0$

Proof

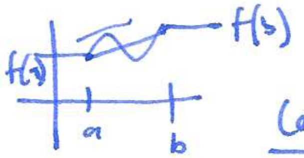


• if there is a local max/min at $c \in (a,b)$ then $f'(c) = 0$

• if no local max/min, then $f(x) = \text{const} = f(a) = f(b)$ so $f'(c) = 0$ for all $c \in (a,b)$ □

§4.3 First derivative test

Thm (MVT) (mean value theorem) suppose f is cts on $[a,b]$ and differentiable on (a,b), then there is a $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$, i.e. there is a point where the slope is equal to the average rate of change.



Proof (Rolle's theorem rotated on its side) □

Corollary If $f(x)$ is differentiable and $f'(x) = 0$ then $f(x) = c$ const.

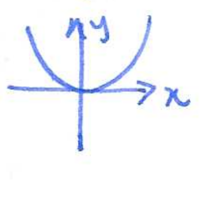
Proof suppose there is $a \neq b$ with $f(a) \neq f(b)$, then there is a $c \in (a,b)$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a} \neq 0$ ※ □.

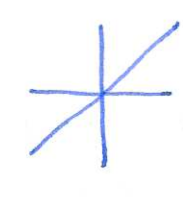
Monotonicity suppose f is differentiable on (a,b):

If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on (a,b)

If $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on (a,b)

Example $f(x) = x^2$ $f'(x) = 2x$ f





increasing on $(0, \infty)$

decreasing on $(-\infty, 0)$