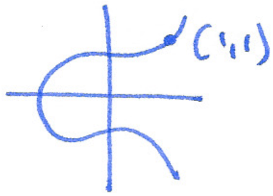
Examples $\frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1}\cdot g'(x)$

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\cdot g'(x)$$

$$\frac{d}{dx}\left(f(ax+b)\right) = af'(ax+b)$$

§3.8 Implicit differentiation

suppose $y^4 + xy = x^3 - x + 2$



don't know how to solve for y explicitly

can think of y as a function of x, $y(x)$

$$(y(x))^4 + x\,y(x) = x^3 - x + 2 \quad \leftarrow \text{differentiate wrt } x \text{ using chain rule}$$

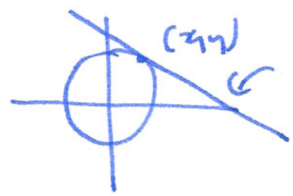
$$4(y(x))^3 y'(x) + y(x) + xy'(x) = 3x^2 - 1$$

$$y'(x)\left(4y^3 + x\right) = 3x^2 - 1 - y$$

$$y' = \frac{3x^2 - 1 - y}{4y^3 + x} \qquad y'(1,1) = \frac{1}{5}$$

Example $x^2 + y^2 = 1$

$2x + 2yy' = 0 \qquad y' = -x/y$



slope $-\frac{x}{y}$

Application: derivatives of inverse functions

Example $y = \ln(x)$

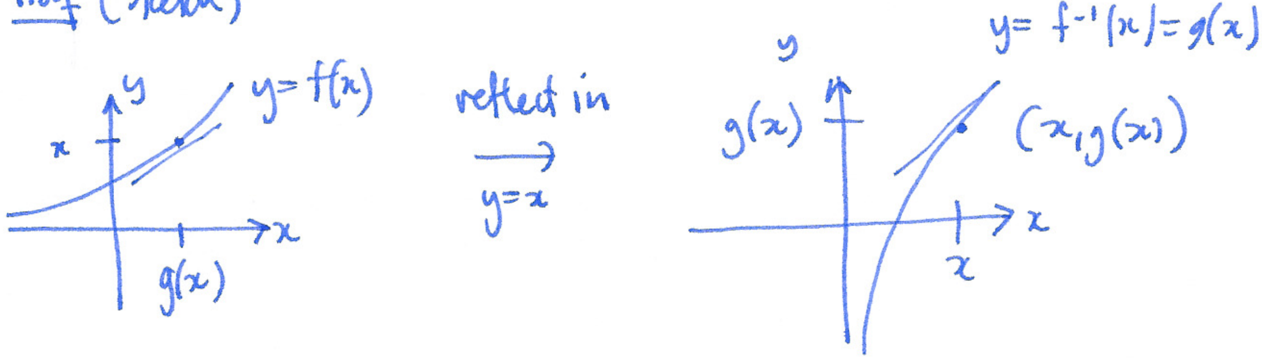
$\boxed{\text{Thm } \frac{d}{dx}(\ln(x)) = \frac{1}{x}}$

$$e^y = x$$

$$e^y \cdot y' = 1 \qquad y' = e^{-y} = e^{-\ln(x)} = \frac{1}{x}$$

Thm If $f(x)$ is differentiable and one-to-one, with inverse $f^{-1}(x) = g(x)$, then $g'(x) = \frac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

Proof (sketch)



$\frac{1}{\text{slope}}$ at $g(x)$, i.e. $\frac{1}{f'(g(x))}$ $\longleftarrow$ slope at $(x, g(x))$: $g'(x)$ $\square$

Application: derivatives of trig functions

Thm $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$

Proof (of $\sin^{-1}(x)$):

$$y = \sin^{-1}(x)$$
$$\sin(y) = x$$
$$\cos(y)\, y' = 1$$
$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \quad \square.$$

Thm $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$ $\qquad \frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1+x^2}$

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\qquad \frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Proof (of $\tan^{-1}(x)$)

$$y = \tan^{-1}(x)$$
$$\tan(y) = x$$
$$\sec^2(y)\, y' = 1$$
$$y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \quad \square$$

Example $\frac{d}{dx}\left(\tan^{-1}(e^{2x})\right) = \frac{1}{1+(e^{2x})^2}\cdot e^{2x}\cdot 2$

§3.9 Derivatives of exponentials and logs

recall: $f(x) = b^x$ then $f'(x) = m_b\, b^x$

$f(x) = e^x$ then $f'(x) = e^x$

Thm $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$

Proof $f(x) = b^x = e^{\ln(b)x}$ $\quad f'(x) = e^{\ln(b)x}\cdot\ln(b) = \ln(b)\, b^x$. $\square$

Example $f(x) = 3^{4x} = (3^4)^x$ $\quad f'(x) = \ln(3^4)(3^4)^x = 4\ln(3)\, 3^{4x}$

Thm $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$ $\square$.

Example ① $f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}$ $\quad f'(x) = \frac{1}{x\ln(b)}$

② $f(x) = x\ln(x)$; $f'(x) = \ln(x) + x\cdot\frac{1}{x} = \ln(x)+1$

Hyperbolic trig functions

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\quad \cosh(x) = \frac{e^x + e^{-x}}{2}$ $\quad$ **note**: $\cosh^2 x - \sinh^2 x = 1$

$\frac{d}{dx}(\sinh(x)) = \cosh x$ $\quad \frac{d}{dx}(\cosh x) = \sinh x$

check $(\sinh(x))' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ so $(\tanh(x))' = \frac{1}{\cosh^2 x}$ etc. ...

inverse functions: $\frac{d}{dx}(\sinh^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}$ $\quad \frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\tanh^{-1}(x)) = \frac{1}{1-x^2}$

② $f(x) = x\ln(x)$, $f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$

Hyperbolic trig functions

$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\qquad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$\frac{d}{dx}(\sinh(x)) = \cosh(x)$ $\qquad \frac{d}{dx}(\cosh(x)) = \sinh(x)$ $\qquad$ note: $\cosh^2 x - \sinh^2 x = 1$

check: $(\sinh(x))' = \left(\dfrac{e^x - e^{-x}}{2}\right)' = \dfrac{e^x + e^{-x}}{2} = \cosh(x)$

$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ so $\frac{d}{dx}(\tanh(x)) = \dfrac{1}{\cosh^2 x}$, etc.

inverse functions: $\frac{d}{dx}(\sinh^{-1}(x)) = \dfrac{1}{\sqrt{x^2+1}}$ $\qquad \frac{d}{dx}(\cosh^{-1}(x)) = \dfrac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\tanh^{-1}(x)) = \dfrac{1}{1-x^2}$

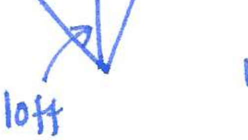
§ 3.10 Related rates

Example ① Ballon  $V = \frac{4}{3}\pi r^3$, inflate ballon $V(t) = \frac{4}{3}\pi (r(t))^3$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ so if $\dfrac{dV}{dt} = 1\,ft^3/min$ and $r = 1ft$, then

$1 = 4\pi 1^2 \dfrac{dr}{dt}$ so $\dfrac{dr}{dt} = \dfrac{1}{4\pi}\,ft/min$

② filling a conical tank



Q: how fast is the water rising when $h = 3ft$?

water in: $\dfrac{dV}{dt} = 10\,ft^3/min$, volume of cone $V = \frac{1}{3}\pi h r^3$

want r in terms of h: $\dfrac{r}{h} = \dfrac{4}{10}$, i.e. $r = \frac{2}{5}h$

so where $V = \frac{1}{3}\pi h \left(\frac{2}{5}h\right)^2 = \frac{4}{75}\pi h^3$, so $\dfrac{dV}{dt} = \dfrac{12}{75}\pi h^2 \dfrac{dh}{dt}$

so when $h = 5$, $10 = \dfrac{12}{75}\pi 5^2 \dfrac{dh}{dt}$, $\dfrac{dh}{dt} = \dfrac{10}{4\pi}\,ft/min$.