Proof (follows from limit laws)

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$$

$$(kf)'(x) = \lim_{h\to 0} \frac{(kf)(x+h) - (kf)(x)}{h} = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h}$$

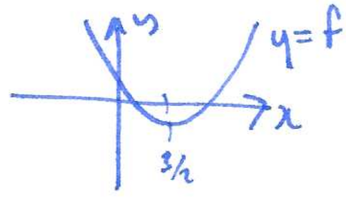
$$= k\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x). \quad \square$$

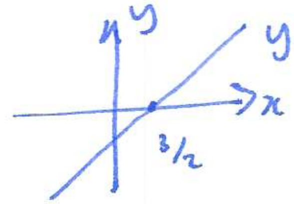
Example $f(x) = x^2 - 3x + 2$. find $f'(x)$

$$\frac{df}{dx} = \frac{d}{dx}(x^2 - 3x + 2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2)$$

$$= 2x - 3\frac{d}{dx}(x) + 0 = 2x - 3.$$

graphs:





Derivative of e^x consider $f(x) = b^x$ $(x>0)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} b^x \frac{b^h - 1}{h}$$

$$= b^x \underbrace{\lim_{h\to 0} \frac{b^h - 1}{1}}$$

this limit doesn't depend on x!
Assume this limit exists and call it m_b

we have shown: for exponential functions, the derivative is proportional to the values of the original function, i.e. if $f(x) = b^x$ then $f'(x) = m_b b^x$. in particular, the slope at $x = 0$ is m_b.

<u>recall</u>: e is defined to be the special number such that the slope of e^x at $x=0$ is equal to 1, therefore $\boxed{\text{if } f(x)=e^x \text{ then } f'(x)=e^x}$ $\frac{d}{dx}(e^x)=e^x$

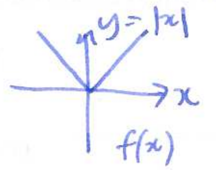
<u>Example</u> $\frac{d}{dx}(7e^x+8x^2) = 7e^x+16x$

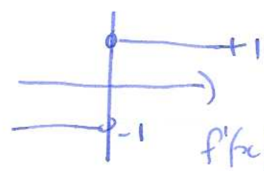
<u>observation</u> this shows that e^x is <u>not</u> a polynomial.

$\frac{d^n}{dx^n}(p(x))$ ← degree goes down by 1 each time, eventually get zero.

<u>Thm</u> Differentiable ⇒ continuous (warning: continuous ⇏ differentiable)

<u>Example</u> $f(x)=|x|$ cts





$f'(x)$ not cts at $x=0$.

<u>claim</u> $f(x)=|x|$ not differentiable at $x=0$.

<u>check</u>: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$

$\lim_{h\to 0+} \frac{|h|}{h} = +1$ $\quad \lim_{h\to 0-} \frac{|h|}{h} = -1$ $\Rightarrow \lim_{h\to 0} \frac{|h|}{h}$ DNE. □

<u>local picture</u> if $f(x)$ is differentiable at $x=c$, means $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. (want to show $\lim_{x\to c} f(x) = f(c)$) consider $f(c+h)-f(c) = h\cdot\frac{f(c+h)-f(c)}{h}$ so $\lim_{h\to 0} f(c+h)-f(c) = \lim_{h\to 0} h\cdot\frac{f(c+h)-f(c)}{h} =$

$\lim_{h\to 0} h \cdot f'(c) = 0$. □.

<u>§ 3.3 Product and quotient rules</u>

new functions from old: $f(x)g(x)$ ← product $\quad \frac{f(x)}{g(x)}$ ← quotient.

<u>Thm</u> (product rule) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

__warning__ : $(fg)' \neq f'g'$!!!

__Examples__ ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x.x) = \frac{d}{dx}(x).x + x.\frac{d}{dx}(x) = x+x = 2x$

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = (3x^2)'(x^2+1) + 3x^2(x^2+1)' = 6x(x^2+1) + 3x^2.2x$

③ $\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$

__Proof__ (of product rule) (assume f, g differentiable at x)

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \cancel{f(x+h)}\,\frac{f(x+h)-f(x)}{h}g(x)$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} g(x)$$

$$= f(x)g'(x) + f'(x)g(x) \quad \square$$

__Thm__ Quotient rule (assume f, g differentiable at x and $g(x) \neq 0$)

$$\left(\frac{f}{g}\right)'(x) = \frac{gf' - fg'}{g^2} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \quad \square$$

__Examples__ ① $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - (x+1)'(x)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

② $\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t)(e^t)' - (e^t+t)'e^t}{(e^t+t)^2} = \frac{(e^t+t)e^t - (e^t+1)e^t}{(e^t+t)^2}$

$$= \frac{te^t - e^t}{(e^t+t)^2}$$