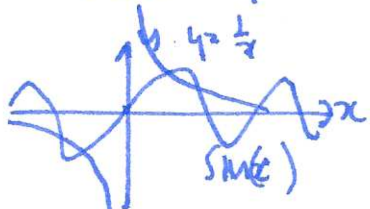
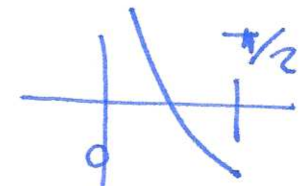
special case : finding zeros

Corollary if $f(x)$ is cts on $[a,b]$ and $f(a)$ and $f(b)$ have different signs, then there is at least one $c \in [a,b]$ s.t. $f(c) = 0$. □

Bisection method find a solution to $\sin x = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin x$

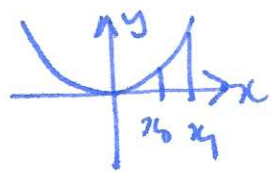




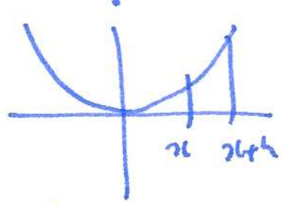
$f(0) = +\infty > 0$ $f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 < 0$, midpoint $f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.565 > 0$, so at least one solution in $[\frac{\pi}{4}, \frac{\pi}{2}]$, etc. ...

§3.1 Defn of the derivative

Recall : we can compute the average rate of change of a function over an interval



$[x_0, x_1]$ $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$



$[x, x+h]$ $\frac{f(x+h) - f(x)}{h}$

Q: how do we find the slope of the tangent line?

A: look at the average rate of change over a small interval $[x, x+h]$ and take the limit as $h \to 0$

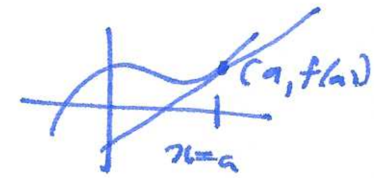
Defn the slope of the tangent line at $x = a$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

notation : also called the derivative at a, written $f'(a)$ or $\frac{df}{dx}(a)$
(Newton) (Leibnitz)

If this limit exists we say that the function is differentiable at $x = a$

Note : $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

<u>Def^{n}</u> the tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$.



The equation for this line is: $y - y_0 = m(x - x_0)$

$$y - f(a) = f'(a)(x-a)$$

or $\quad y = f(a) + f'(a)(x-a)$

<u>Example</u> find the tangent line to $y = x^2$ at $x = 1$

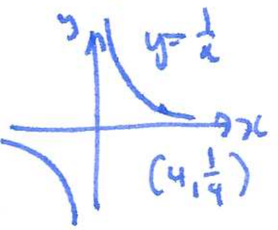
$(x, f(x)) = (1,1)$ slope $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0} \dfrac{(1+h)^2 - 1}{h}$

$= \lim\limits_{h\to 0} \dfrac{1 + 2h + h^2 - 1}{h} = \lim\limits_{h\to 0} 2+h = 2$

so equation of tangent line is $\quad y - 1 = 2(x-1)$ $\quad y = 1 + 2(x-1)$

$y = 2x - 1$

<u>Example</u> find the slope of the tangent line to $y = \frac{1}{x}$ at $x = 4$



slope $f'(4) = \lim\limits_{h\to 0} \dfrac{f(4+h) - f(4)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1}{4+h} - \frac{1}{4}}{h}$

$= \lim\limits_{h\to 0} \frac{1}{h}\left(\frac{1}{4+h} - \frac{1}{4}\right) = \lim\limits_{h\to 0} \dfrac{4 - (4+h)}{h(4+h)4} = \lim\limits_{h\to 0} \dfrac{-h}{h(4+h)4}$

$= \lim\limits_{h\to 0} \dfrac{-1}{4(4+h)} = \dfrac{-1}{16}$ $\quad$ tangent line: $\quad y - \frac{1}{4} = -\frac{1}{16}(x-4)$

<u>Example</u>: find slope of tangent line to $f(x) = mx + b$ at $x = a$:

$f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{m(a+h) + b - (ma+b)}{h} = \lim\limits_{h\to 0} \dfrac{mh}{h}$

$= \lim\limits_{h\to 0} m = m$ $\qquad$ <u>Observation</u>: if $f(x) = b$ (constant) then $f'(x) = 0$ for all x.

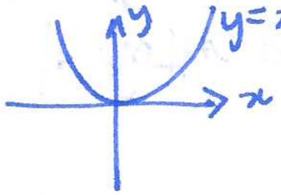
<u>§ 3.2 Derivative as a function</u> at each point x, there is a tangent line, with a slope, which is a number. Therefore, we can define a

$\mathbb{R} \to \mathbb{R}$

$x \mapsto$ slope of tangent line at $(x, f(x))$

__notation__: we call this function $f'(x)$ "the derivative of f"

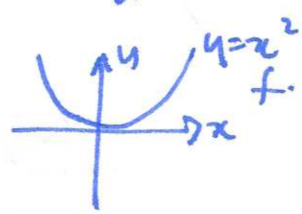
__Example__

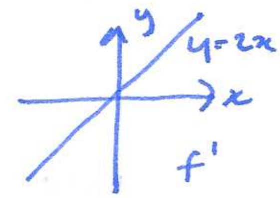


then $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ with $f(x) = x^2$

then $f'(x) = \lim\limits_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim\limits_{h \to 0} 2x + h = 2x$

__summary__: if $f(x) = x^2$, then $f'(x) = 2x$

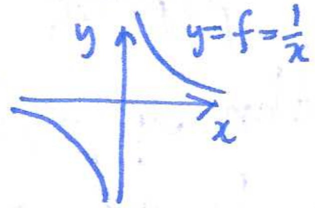


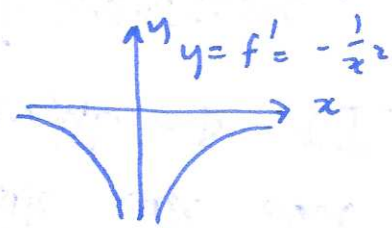


__Example__ $f(x) = \frac{1}{x}$

$f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim\limits_{h \to 0} \frac{x - (x+h)}{h(x+h)x}$

$= \lim\limits_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$





__Remarks__ ① functions f : domain $\to$ range, eg $f : \mathbb{R} \to \mathbb{R}$

derivative : (differentiable) functions $\to$ functions

$f \longmapsto f'$

warning: not all functions differentiable.

② "calculus" means rules for doing calculations. We don't have to explicitly compute limits all the time.

__Example__ $f(x) = x^3$ $f'(x) = \lim\limits_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim\limits_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim\limits_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$

<u>Thm</u> (powers of x) if $f(x) = x^n$ then $f'(x) = nx^{n-1}$
(works for all $n \in \mathbb{R}$!)

<u>Examples</u> $\frac{d}{dx}(x^2) = 2x$ $\quad \frac{d}{dx}(x^3) = 3x^2$ $\quad \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$

$\frac{d}{dx}(x^1) = 1$ $\quad \frac{d}{dx}(x^0) = 0$ $\quad \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

<u>Thm</u> $\frac{d}{dx}(x^n) = nx^{n-1}$

<u>Proof</u> $f(x) = x^n$ $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to x} \frac{(x+h)^n - x^n}{h}$

binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n-1}xh^{n-1} + h^n$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

so $\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + h^2(-) - x^n}{h} = \lim_{h \to 0} nx^{n-1} + h(-)$

$= nx^{n-1}$ $\square$

<u>warning</u>: this rule only works for polynomials <u>not</u> exponentials.

$f(x) = x^{100}$ $\quad f'(x) = 100x^{99}$ $\qquad f(x) = 2^x$ <u>not</u> polynomial.

<u>other useful rules</u>

<u>Thm</u> (linearity) If f and g are differentiable functions, then:

- $f+g$ is differentiable with $(f+g)' = f' + g'$
  $\iff \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$
- k constant $(kf)' = kf'$ $\iff \frac{d}{dx}(kf) = k\frac{df}{dx}$