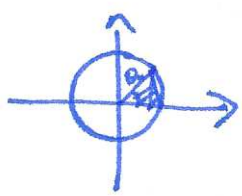
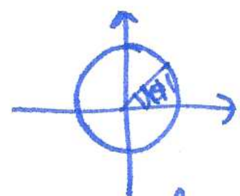
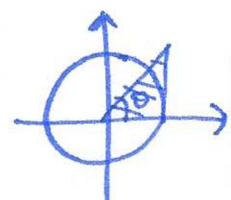
$-1 \leq \sin(x) \leq 1$

$-|x| \leq x\sin(x) \leq |x|$ $\quad \lim\limits_{x\to} |x| = 0 \quad \lim\limits_{x\to 0} -|x| = 0 \Rightarrow \lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

$\underline{\text{Thm}}$ $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 \qquad \lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$

$\underline{\text{Proof}}$ (of $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$, consider the following three areas:

area of triangle $\frac{1}{2}bh$ $\quad \frac{1}{2}\cdot 1 \cdot \sin\theta$

area of sector $\pi r^2 \cdot \frac{\theta}{2\pi}$

area of triangle $\frac{1}{2}bh = \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

regions subsets of each other:

$$\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \leq 1 \qquad \cos\theta \leq \frac{\sin\theta}{\theta}$$

so

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

$\lim\limits_{\theta\to 0} \cos\theta = 1 \qquad \lim\limits_{\theta\to 0} 1 = 1$ $\qquad$ squeeze thm $\Rightarrow$ $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

$\underline{\text{Examples}}$ ① $\lim\limits_{x\to 0} \dfrac{\sin 3x}{2x}$ $\qquad$ know: $\lim\limits_{x\to 0} \dfrac{\sin\theta}{\theta} = 1$

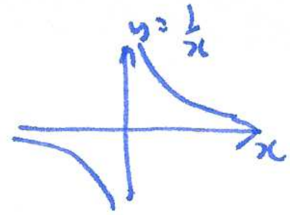
write $3x = \theta$

$$\lim_{\theta\to 0} \frac{\sin\theta}{2\cdot\theta/3} = \lim_{\theta\to 0} \frac{3}{2}\frac{\sin\theta}{\theta} = \frac{3}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{3}{2}$$

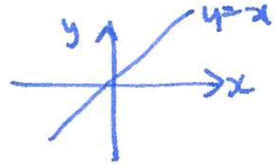
②
$$\lim_{t\to 0}\frac{1-\cos t}{\sin t} = \lim_{t\to 0}\frac{1-\cos t}{t}\cdot\frac{t}{\sin t} = \underbrace{\lim_{t\to 0}\frac{1-\cos t}{t}}_{0}\cdot\underbrace{\lim_{t\to 0}\frac{t}{\sin t}}_{1} = 0$$

§2.7 Limits at infinity

key observations: $\lim\limits_{x \to 0} \frac{1}{x} = 0$



$\lim\limits_{x \to \infty} x = +\infty$



<u>Examples</u> ① $\lim\limits_{x \to \infty} \frac{3x}{2x-1} = \lim\limits_{x \to \infty} \frac{3}{2 - 1/x} = \frac{3}{2}$

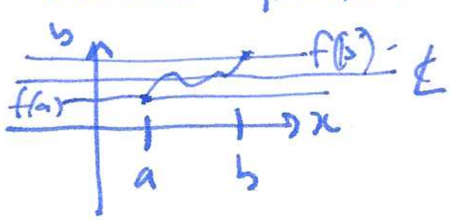
② $\lim\limits_{x \to \infty} \frac{x^2+x}{x-3} = \lim\limits_{x \to \infty} \frac{x+1}{1 - 3/x} = +\infty$

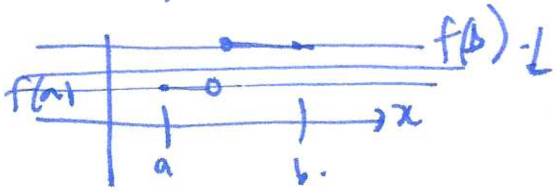
③ $\lim\limits_{x \to \infty} \frac{1}{x} - \frac{2}{3x+1} = \lim\limits_{x \to \infty} \frac{1}{x} - \lim\limits_{x \to \infty} \frac{3}{3x+1} = 0 - 0 = 0$

④ $\lim\limits_{x \to 0} \frac{\sqrt{2x^2+1}}{4x+1} = \lim\limits_{x \to \infty} \frac{\sqrt{2 + 1/x^2}}{4 + 1/x} = \frac{\sqrt{2}}{4}$ (do $\lim\limits_{x \to -\infty}$!)

§2.8 Intermediate value theorem (IVT)

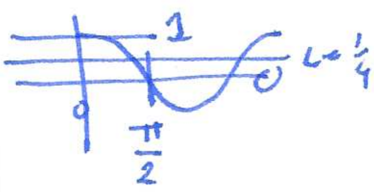
"continuous functions can't skip values"





<u>Thm</u> (IVT) If $f(x)$ is a cts function on a closed interval $[a,b]$ with $f(a) \neq f(b)$ then for any number L with $f(a) \leq L \leq f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = L$ $\square$

<u>Example</u> show $\cos(x) = \frac{1}{4}$ has at least one solution.



consider $[0, \frac{\pi}{2}]$ $\cos(0) = 1$ $\cos(\frac{\pi}{2}) = 0$ $0 \leq \frac{1}{4} \leq 1$

IVT $\Rightarrow$ there is $c \in [a,b]$ s.t. $f(c) = \frac{1}{4}$.