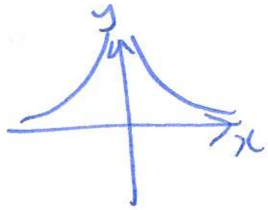
<u>example</u> $f(x) = \dfrac{x}{|x|}$ $\quad \left.\begin{array}{l} \lim\limits_{x\to 0+} f(x) = +1 \\ \lim\limits_{x\to 0-} f(x) = -1 \end{array}\right\} \; 1 \neq -1$ so $\lim\limits_{x\to 0} f(x)$ DNE.

<u>Example</u> $f(x) = \dfrac{1}{x^2}$



$$\left.\begin{array}{l} \lim\limits_{x\to 0+} \dfrac{1}{x^2} = +\infty \\ \lim\limits_{x\to 0-} \dfrac{1}{x^2} = +\infty \end{array}\right\} \lim_{x\to 0} \frac{1}{x^2} = +\infty$$

§2.3 Basic limit laws

<u>Example</u> $\lim\limits_{x\to 0} 2x+2 = \lim\limits_{x\to 0} 2x + \lim\limits_{x\to 0} 2 = 2\lim\limits_{x\to 0} x + \lim\limits_{x\to 0} 2 = 2$

<u>Thm</u> assume that $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist and are finite. Then:

1) sums: $\lim\limits_{x\to c} f(x)+g(x) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$

2) constant multiple: $\lim\limits_{x\to c} kf(x) = k \lim\limits_{x\to c} f(x)$ $\quad k$ constant (does not depend on x)

3) products: $\lim\limits_{x\to c} (f(x)\,g(x)) = (\lim\limits_{x\to c} f(x))(\lim\limits_{x\to c} g(x))$

4) quotients: $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$ as long as $\lim\limits_{x\to c} g(x) \neq 0$

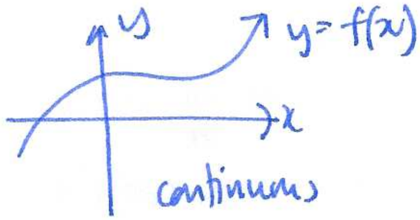
<u>warning</u>: these rules don't work if either $\lim\limits_{x\to c} f(x) = \text{DNE}$ or $\lim\limits_{x\to c} g(x) = \text{DNE}$.

<u>Examples</u> $\lim\limits_{x\to 3} x^2 = \left(\lim\limits_{x\to 3} x\right)\left(\lim\limits_{x\to 3} x\right) = 3.3 = 9$

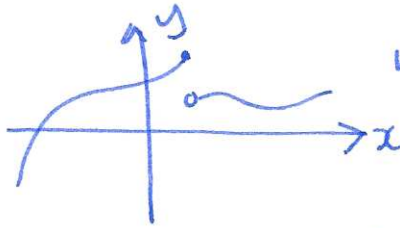
$$\lim_{t\to 2} \frac{t+5}{2t} = \frac{\lim\limits_{t\to 2} t+5}{\lim\limits_{t\to 2} 2t} = \frac{\left(\lim\limits_{t\to 2} t\right)+5}{2\left(\lim\limits_{t\to 2} t\right)} = \frac{7}{4}.$$

§2.4 Limits and continuity

Example



continuous

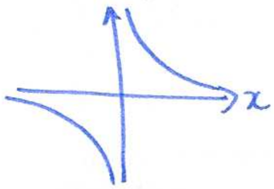


not continuous / discontinuous

Defn We say $f(x)$ is continuous at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

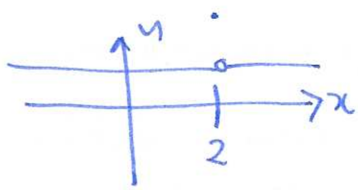
If the limit is DNE, infinite, or not equal to $f(c)$ then $f(x)$ is not continuous at $x=c$.

Examples



$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & \text{at } x = 0 \end{cases}$$

← not cts at $x=0$



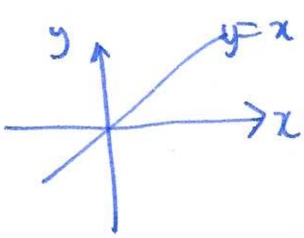
$$f(x) = \begin{cases} 1 & x \neq 2 \\ 2 & x = 2 \end{cases}$$

$\lim_{x\to 2} f(x) = 1$

but $f(2) = 2$

not cts at $x=2$

Example show $f(x) = x$ is cts.



$f(c) = c$, want to show $\lim_{x\to c} f(x) = c$

follows from limit laws: $\lim_{x\to c} x = c$ ✓.

Corollary: polynomials / rational functions are continuous where defined

Defn $f(x)$ is left continuous at $x=c$ if $\lim_{x\to c^-} f(x) = f(c)$

right continuous at $x=c$ if $\lim_{x\to c^+} f(x) = f(c)$

If at least one of the right or left limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x=c$

Building continuous functions

Thm 0 $f(x) = k$, $f(x) = x$ are continuous.

Thm 1 suppose $f(x), g(x)$ are continuous at $x = c$, then the following functions are continuous at $x = c$:

1) $f(x) + g(x)$ 2) $kf(x)$ k constant

3) $f(x)g(x)$ 4) $\frac{f(x)}{g(x)}$ if $g(c) \neq 0$

Proof these follow immediately from the limit laws.

check: 1) $f(x)$ continuous means $\lim_{x \to c} f(x) = f(c)$

$g(x)$ " $\lim_{x \to c} g(x) = g(c)$

so $\lim_{x \to c} (f(x) + g(x)) = (\lim_{x \to c} f(x)) + (\lim_{x \to c} g(x)) = f(c) + g(c)$

as required $\square$

Thm 2 Polynomials are cts. $p(x) = a_n x^n + \dots + a_0$

Rational functions $\frac{p(x)}{q(x)}$ are continuous where $q(x) \neq 0$

Proof $f(x) = x$ is continuous. so $f(x)f(x) = x^2$ is cts. (products)

so $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is cts. (multiplication by constants and sums)

So $\frac{p(x)}{q(x)}$ is cts (quotients) where $q(x) \neq 0$. $\square$.

Useful facts

Thm 3.
- $\sin(x), \cos(x)$ are cts
- $x^{1/n}$ is cts.
- b^x is cts
- $\log_b(x)$ is cts

[combination of these with polynomials sometimes called elementary functions]