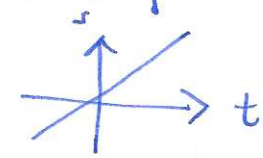
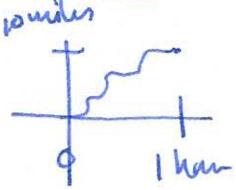
§2.1 Limits, rates of change, tangent lines

motivation: velocity $= \dfrac{\text{distance}}{\text{time}}$ example: driving at constant speed.
velocity = slope of line

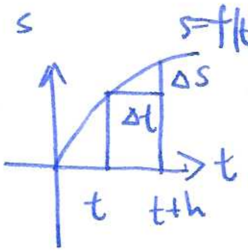


problem: what happens if you don't drive at constant speed?



average speed $= \dfrac{\text{distance travelled}}{\text{time}}$

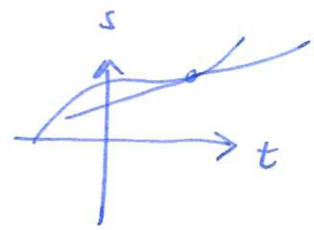
- we can look at average speed over any time interval, including very short ones.



average speed on interval $[t, t+h]$ is

$$\frac{\Delta s}{\Delta t} = \frac{f(t+h) - f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$$

Q: what is the speed at time t?
(sometimes called the instantaneous speed / rate of change)



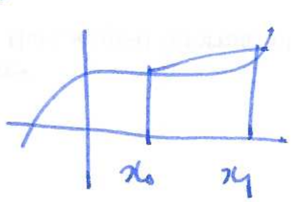
A: the speed (velocity) is the slope of the tangent line at t

idea / hope: as the length of the interval $[t, t+h]$ gets small, the average speed gets closer to the slope of the tangent line.
this works for "nice" functions

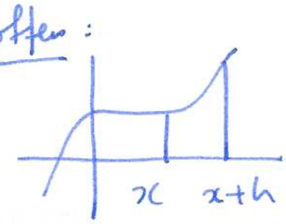
observation: this works for any function $y = f(x)$, not just distance

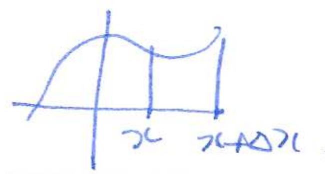
summary: average rate of change over an interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



often:

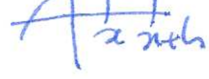




§2.2 Limits

aim: want to find slopes of tangent lines

know: average rate of change $\dfrac{f(x+h) - f(x)}{h}$



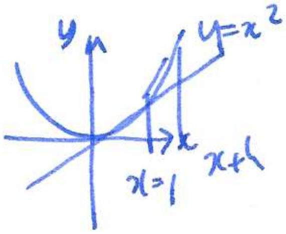
Q: why not just set $h = 0$?

A: doesn't work, get $\dfrac{f(x) - f(x)}{0} = \dfrac{0}{0}$ undefined ☹

Observations

① if we draw careful pictures, the average slope gets close to the slope of the tangent line as the length of the interval gets small.

② seems to work for sample calculation



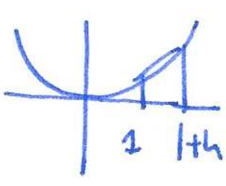
$h = 1$: $\dfrac{f(2) - f(1)}{2 - 1} = \dfrac{4 - 1}{1} = 3$

$h = 0.5$ $\dfrac{f(1.5) - f(1)}{1.5 - 1} = \dfrac{\frac{9}{4} - 1}{\frac{1}{2}} = \dfrac{5}{2} = 2.5$

$h = 0.1$ $\dfrac{f(1.1) - f(1)}{1.1 - 1} = \dfrac{1.21 - 1}{0.1} = 2.1$

$h = 0.01$ $\dfrac{1.0201 - 1}{0.01} = 2.01$

③ seems to work algebraically

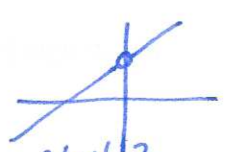


average rate of change from $x = 1$ to $x = 1 + h$ is

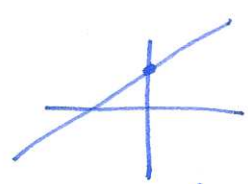
$$\frac{f(1+h) - f(1)}{1+h-1} = \frac{(1+h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h}$$

$= 2 + h$
↑
$h \neq 0$!

warning



$\dfrac{2h+h^2}{h}$



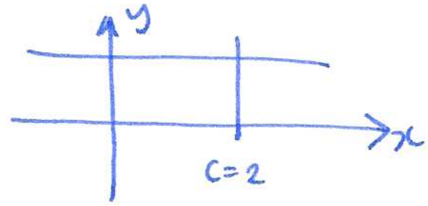
$2 + h$

Def^n Let f be a function defined on an interval containing c, but not necessarily at c. We say "the limit of $f(x)$ as x approaches c is equal to L" if $|f(x) - L|$ becomes arbitrarily small as x gets close to c.

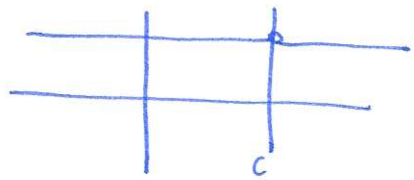
notation $\lim\limits_{x \to c} f(x) = L$ ~~&~~ or $f(x) \to L$ as $x \to c$

we also say "$f(x)$ converges to L as x tends to c".

Examples a) $f(x) = 5$, $c = 2$



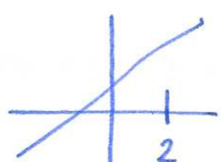
b) $f(x) = \frac{5(x-2)}{(x-2)}$, $c = 2$



want to show $|f(x) - 5|$ close to 0 if x close to ~~&~~ 2

$|f(x) - 5| = |5-5| = 0$ for all $x \neq 2$, so this is true.

c) $\lim\limits_{x \to 2} 2x+1 = 5$



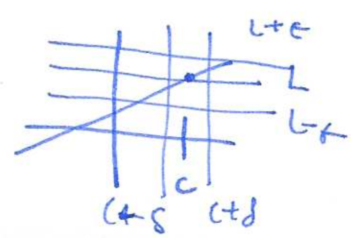
d) $\lim\limits_{x \to 2} \frac{(2x+1)(x-2)}{(x-2)} = \frac{2x^2-3x-2}{x-2}$

want to show $|f(x) - 5|$ close to 0 when x close to 2

$|f(x) - 5| = |2x+1-5| = |2x-4| = 2|x-2|$ x close to 2

$\Leftrightarrow$ $|x-2|$ close to 0.

Precise def$^{\underline{n}}$ let $f(x)$ be defined on an interval containing c, but not necessarily at c. We say $\lim\limits_{x \to c} f(x) = L$ if for all $\epsilon > 0$ there is a $\delta > 0$ s.t. if $|c - x| < \delta$ then $|f(x) - L| < \epsilon$.



useful facts

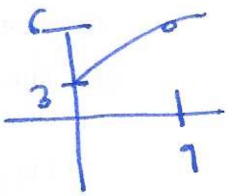
Th$^{\underline{m}}$ for any constants k, c: $\lim\limits_{x \to c} k = k$

$\lim\limits_{x \to c} x = c$

investigating limits : try
- drawing a picture
- calculating close values
- algebra

Example $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$ problem : plug in $x = 9$, get $\frac{0}{0}$ undefined

- draw picture



looks like $f(9) = 6$

- calculate :

x	$\frac{x-9}{\sqrt{x}-3}$
8.9	5.983
9.1	6.016
8.99	5.998
9.01	6.002

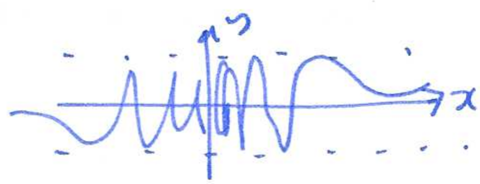
- algebra : difference of two squares

$$x - 9 = (\sqrt{x})^2 - (3)^2 = (\sqrt{x}-3)(\sqrt{x}+3)$$

$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = \sqrt{x}+3 \quad (x \neq 9)!$$

so $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \to 9} \sqrt{x}+3 = 6$

Bad example : no limit $f(x) = \sin(\frac{1}{x})$

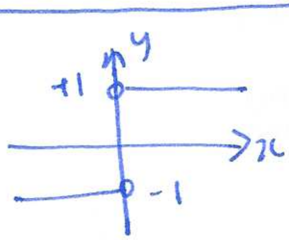


no limit at $x = 0$

note : $f(\frac{1}{2\pi n}) = \sin(2\pi n) = 0$

$f(\frac{1}{2\pi n + \frac{\pi}{2}}) = \sin(2\pi n + \frac{\pi}{2}) = 1$

One side limits example $f(x) = \frac{x}{|x|}$ $\quad f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$



sometimes useful to distinguish left/right/two sided limits

notation $\lim_{x \to 0+} f(x)$ means right limit ($x > 0$)

$\lim_{x \to 0-} f(x)$ means left limit ($x < 0$)

note : two sided limit exists if and only if left and right limits exist and are equal.