<u>last time</u>: homotopy $F: X\times I\to Y$ $f\simeq g$ $F(x,0)=f(x)$, $F(x,1)=g(x)$ 3/4 ①

- homotopy equivalence $X\simeq Y$ if $\exists\ X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $gf\simeq \mathbb{1}_X$ $fg\simeq \mathbb{1}_Y$

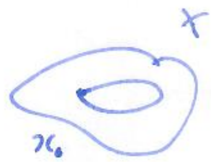
$\Rightarrow H_k(X)=H_k(Y)$.

- X contractible if $X\simeq\{pt\}$
- path components of X $\Rightarrow$. $x \xrightarrow{f} y \xrightarrow{g} z$ defined <u>path composition</u> $f\cdot g: I\to X$

<u>Recall</u> a group G is a set, with an operation $G\times G\to G$ s.t.

- ~~inverses~~ identity: $\exists\, e$ s.t. $ge=eg=g$
- inverse $\forall g\ \exists\, g^{-1}$ s.t. $gg^{-1}=g^{-1}g=e$
- associative $(fg)h=f(gh)$

<u>not nec abelian!</u>



<u>Fundamental group</u> idea: make groups out of loops in X.

<u>setup</u>: pick basepoint $x_0\in X$ <u>Defn</u> a <u>loop</u> based at x is a path $f: I\to X$ st. $f(0)=f(1)=x$

Q: what is group operation? A: want it to be $f\cdot g$

Q: what is identity? A: $e: I\to x$ constant loop.

<u>Problem</u> $e\cdot f\neq f$ as loops

<u>Solution</u> say two loops are equivalent if they are homotopic <u>rel basepoint</u>

<u>Defn</u> $F: I\times I\to I$ is a homotopy rel basepoints between loops f,g if

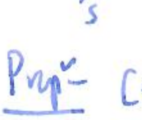
$F(t,0)=f(t)$, $F(t,1)=g(t)$, $F(0,s)=x$, $F(1,s)=x$

<u>Claim</u> this is an equivalence relation on based loops.

<u>Proof</u> reflexive: clear $F(s,t)=f(s)$

symmetric: if $F: I\times I\to I$ is a homotopy rel endpoints from f to g, then $\bar F: I\times I\to I$ $\bar F(s,t)=F(s,1-t)$ is a homotopy rel endpoints from g to f.

associativity: F homotopy from f to g define $H: I\times I\to X$
G homotopy from g to h

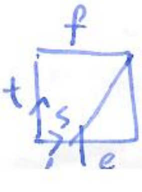


$$H(s,t)=\begin{cases} F(s,2t) & 0\le t\le \frac12 \\ G(s,2t-1) & \frac12\le t\le 1\end{cases}$$ $\square$.

<u>Prop</u> $[e][f]=[e\cdot f]=[f]$

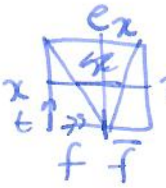
$$F(s,t)=\begin{cases} x & \text{if } s\le 1-2t \\ f(s(2t-1)+1-2t) & s\ge 1-2t\end{cases}$$

$[f][e]=[f\cdot e]=[f]$



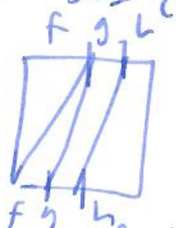
$$F(s,t)=\begin{cases} x & \text{if } s\ge 2t-1 \\ f((2t-1)s) & \dots\end{cases}$$

inverse defn $\bar f(s) = f(1-s)$ then Prop: $[f][\bar f] = [e]$

Proof  

$$F(s,t) = \begin{cases} f(s) & s \le 1-2t \\ f(1-2t) & 1-2t \le s \le 2t-1 \\ \bar f(s) & s \ge 2t-1 \end{cases}$$ □

Prop: associative: $(fg)h = f(gh)$. Proof



□.

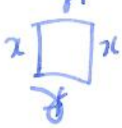
Conclusion $(X,x) \rightsquigarrow \pi_1(X,x) \leftarrow$ fundamental group of X.

Prop: $f: X \to Y$ cts, $x \mapsto y$ gives homomorphism ~~map~~ $f_*: \pi_1(X,x) \to \pi_1(Y,y)$

Proof send $[\gamma] \mapsto [f\circ\gamma]$

check: well defined: suppose $\gamma \simeq \gamma'$ then $f\circ\gamma \simeq f\circ\gamma'$

$F: I\times I \to X$ $f\circ F: I\times I \to X$

  ✓

check: homomorphism: want:
$$[f\circ\gamma_1][f\circ\gamma_2] = [f\circ(\gamma_1\cdot\gamma_2)]$$

Proof $[f\circ\gamma_1][f\circ\gamma_2] = [(f\circ\gamma_1)\cdot(f\circ\gamma_2)] = [f\circ(\gamma_1\cdot\gamma_2)]$ □.

Example if X contractible $\pi_1(X,x_0) = 1$.

Proof $X \underset{g}{\overset{f}{\rightleftarrows}} x_0$ s.t. $gf \simeq 1\!\!1_X$, i.e. $\exists\, F: X\times I \to X$

s.t. $F(x,0) = gf(x) = x_0$ let $\gamma: I \to X$, $s \mapsto \gamma(s)$ be a path
$F(x,1) = x$ defn $G: I\times I \to X$.

homotopy from γ to e. □. $G(s,t) = F(\gamma(s),t)$

Examples $\pi_1(S^1) = \mathbb{Z}$ $\pi_1(S^1\times S^1) = \mathbb{Z}\oplus\mathbb{Z}$ $\pi_1(S^1\vee S^1) = F_2$