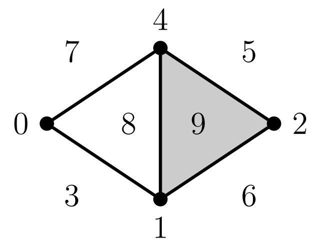
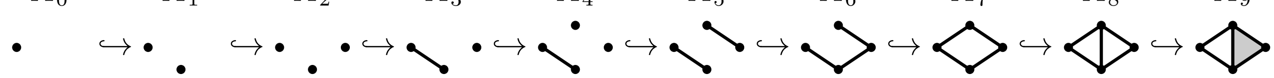
Assignment for week 3

1. Consider the following sequence of simplicial complexes, where the numbers indicate the order in which the simplices are added.



That is:

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow X_4 \hookrightarrow X_5 \hookrightarrow X_6 \hookrightarrow X_7 \hookrightarrow X_8 \hookrightarrow X_9$$



Compute the dimensions of the inclusions maps $i_*\colon H_0(X_j) \to H_0(X_k)$ and $i_*\colon H_1(X_j) \to H_1(X_k)$, for $j \leq k$ and then draw the barcodes for persistent homology.

H_0	0	1	2	3	4	5	6	7	8	9
0										
1	–									
2	–	–								
3	–	–	–							
4	–	–	–	–						
5	–	–	–	–	–					
6	–	–	–	–	–	–				
7	–	–	–	–	–	–	–			
8	–	–	–	–	–	–	–	–		
9	–	–	–	–	–	–	–	–	–	

H_1	0	1	2	3	4	5	6	7	8	9
0										
1	–									
2	–	–								
3	–	–	–							
4	–	–	–	–						
5	–	–	–	–	–					
6	–	–	–	–	–	–				
7	–	–	–	–	–	–	–			
8	–	–	–	–	–	–	–	–		
9	–	–	–	–	–	–	–	–	–	

2. Adapt the first R example to choose points on the unit sphere $S^2 \subset \mathbb{R}^3$. What barcodes would you expect to get in each dimension? Is this what happens? Hint: If you sample x, y and z from three independent normal distributions, then the collection of normalized vectors is uniformly distributed on the sphere, as the three dimensional Gaussian is rotationally symmetric.

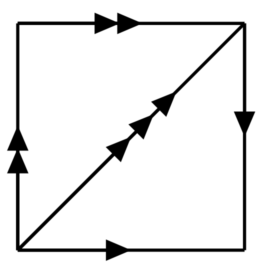
3. Let X_r be the Rips complex of a collection of points $\{x_i\} \subset \mathbb{R}^d$. Let X_r and X_s, with $s > r$ be a pair of Rips complex which differ by the addition of a single edge, and whatever higher dimensional simplices that creates.

   (a) Show that the dimension of $H_0(X_s)$ is the same as the dimension of $H_0(X_r)$, or goes down by exactly one.

   (b) Is there any bound on the change in dimension of H_k for $k = 1$? You don't need to give a detailed proof, just explain why or why not.

4. Use fundamental group to explain why the torus $X = S^1 \times S^1$ is not homotopy equivalent to $Y = S^2 \vee S^1 \vee S^1$, even though they have the same homology groups.

5. Use the following Δ-complex structure on $\mathbb{RP}^2$ to write down a presentation of the fundamental group. Show that $\pi_1(\mathbb{RP}^2)$ is isomorphic to $\mathbb{Z}_2$.



6. Use van Kampen's Theorem to write down a presentation of $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. Show that the abelianization of this group is isomorphic to the homology group $H_1(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Explain why $\pi_1(X)$ is infinite by constructing an explicit infinite order element. You don't need to give a detailed proof of why the element is infinite order.