Math 301 Introduction to Proof Spring 22 Midterm 2

Name: ______________________________

(1) (a) Prove that for all integers n, if n^2 is even, then n is even.
(b) Prove that $\sqrt{2}$ is irrational. (You may use part (a).)

(2) Give examples of:
(a) a function $f \colon \mathbb{R} \to \mathbb{R}$ which is injective but not surjective.
(b) a function $f \colon \mathbb{N} \to \mathbb{Z}$ which is surjective but not injective.

(3) What do the following statements mean in ordinary language?
(a) $(\forall x \in \mathbb{R})(\exists a \in A)(a > x)$
(b) $(\exists x \in \mathbb{R})(\forall a \in A)(a > x)$
(c) $(\exists A \subseteq \mathbb{R})(\forall x \in \mathbb{R})(\exists a \in A)(a > x)$

(4) State the negation of the following statements, using appropriate quantifiers:
(a) The function $f \colon \mathbb{R} \to \mathbb{R}$ is strictly increasing.
(b) The natural number n is prime.

(5) Write out a careful proof or give a counterexample to the following statement:
Let $f \colon X \to Y$ be a function, and let $A, B \subseteq X$. Then $f(A \cap B) \subseteq f(A) \cap f(B)$.

(6) Write out a careful proof or give a counterexample to the following statement:
Let $f \colon X \to Y$ be a function, and let $A, B \subseteq X$. Then $f(A) \cap f(B) \subseteq f(A \cap B)$.

(7) Write out a careful proof or give a counterexample to the following statement:
If $g \circ f$ is surjective then g is surjective.

(8) Write out a careful proof or give a counterexample to the following statement:
If $g \circ f$ is injective then g is injective.

(9) Consider the statement:
Let $f \colon X \to Y$ be a function, and let $A \subseteq X$. If $x \notin A$ then $f(x) \notin f(A)$.
(a) State the contrapositive of the statement, and then prove or give a counterexample.
(b) State the converse of the statement, and then prove or give a counterexample.

(10) Prove or give a counterexample to the following statement:
Let $f \colon X \to Y$ be a function. For any $A, B \subseteq Y$, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

MT2 Solutions

Q1 a) Thm If n^2 ~~is~~ even, then n is even.

Proof (contrapositive) assume n is odd. Then $n = 2k+1$ for some integer k, so $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$, which is odd, so n^2 is odd $\square$.

b) Thm ~~Supp~~ $\sqrt{2}$ is irrational.

Proof (by contradiction) Suppose $\sqrt{2} = a/b$ in lowest terms. Then $2 = a^2/b^2$, so $2b^2 = a^2$. Then a^2 is even, so a is even by a). If a is even, then $a = 2c$ for some $c \in \mathbb{Z}$, so $2b^2 = (2c)^2 = 4c^2$. This implies that $b^2 = 2c^2$, so b^2 is even. Again by a), this implies b is even, but then a and b are both even, so have common factor ~~#~~ a/b lowest terms. $\square$.

Q2 a) $f: \mathbb{R} \to \mathbb{R}$ injective but not surjective : e^x.

b) $f: \mathbb{N} \to \mathbb{Z}$ surjective but injective $f(n) = \begin{cases} \frac{n}{2} - 1 & n \text{ even} \\ -\frac{(n-1)}{2} & n \text{ odd} \end{cases}$

Q3 a) $A \subseteq \mathbb{R}$ not bounded above

b) A is bounded below

c) There is a subset of $\mathbb{R}$ which is not bounded above.

Q4 a) $f: \mathbb{R} \to \mathbb{R}$ strictly increasing: $(\forall x, y \in \mathbb{R})(x > y \Rightarrow f(x) > f(y))$

negation: $(\exists x, y \in \mathbb{R})(x > y \text{ and } f(x) \leq f(y))$.

b) n is prime: $(\forall p)\, p \mid n \Rightarrow (p = 1) \text{ or } (p = n))$

negation: $(\exists p)\, p \mid n$ and $(p \neq 1)$ and $(p \neq n)$

} where $p \in \mathbb{N}$.

Q5 Thm $f(A \cap B) \subseteq f(A) \cap f(B)$.

Proof Suppose $y \in f(A \cap B)$, then there is an $x \in A \cap B$ such that $f(x) = y$. As $x \in A$, this ~~implies~~ implies $f(x) \in f(A)$, as $x \in B$, this implies $f(x) \in f(B)$, therefore $f(x) \in f(A) \cap f(B)$, so $y \in f(A) \cap f(B)$, as required. $\square$.

Q6 $f(A) \cap f(B) \subseteq f(A \cap B)$ false: $1 \to 3$, $2 \to 3$

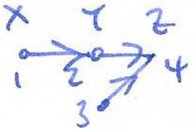
$A = \{1\}$ $f(A) = \{3\}$ $A \cap B = \emptyset$

$B = \{2\}$ $f(B) = \{3\}$ $f(\emptyset) = \emptyset$

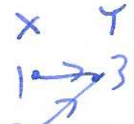
$f(A) \cap f(B) = \{3\} \not\subseteq \emptyset$

Q7 Thm $f: X \to Y$, $g: Y \to Z$, $g \circ f$ surjective implies g is surjective. ②

Proof Assume $g \circ f$ is surjective, then for all $z \in Z$, there is an $x \in X$ s.t. $g(f(x)) = z$ but $f(x) \in Y$, so for all $z \in Z$, there is an element $f(x) \in Y$ st. $g(f(x)) = z$, so g is surjective. $\square$.

Q8 False  $g \circ f$:  injective, but $g(2) = g(3) = 4$.

Q9 a) $f: X \to Y$ function $A \subseteq X$

contrapositive: $f(x) \in f(A)$ then $x \in A$. False:  $A = \{1\}$ $f(\{1\}) = \{3\}$
$f(2) = 3 \in \{3\}$
but $2 \notin A$.

~~Proof~~ converse:

b) $f(x) \notin f(A) \Rightarrow x \notin A$

contrapositive of converse: Thm $x \in A \Rightarrow f(x) \in f(A)$ Proof follows from definition of image:
$f(A) = \{f(x) \mid x \in A\}$ so if $x \in A$, $f(x) \in f(A)$ $\square$.

Q10 $f: X \to Y$, $A, B \subseteq Y$ Thm $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Proof $\subseteq$: suppose $x \in f^{-1}(A \cup B)$, then $f(x) \in A \cup B$, so $f(x) \in A$ or $f(x) \in B$.
This means $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$, so $x \in f^{-1}(A) \cup f^{-1}(B)$ $\square$.

$\supseteq$: suppose $x \in f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. This means $f(x) \in A$, or $f(x) \in B$, so $f(x) \in A \cup B$. This means $x \in f^{-1}(A \cup B)$, as required $\square$.