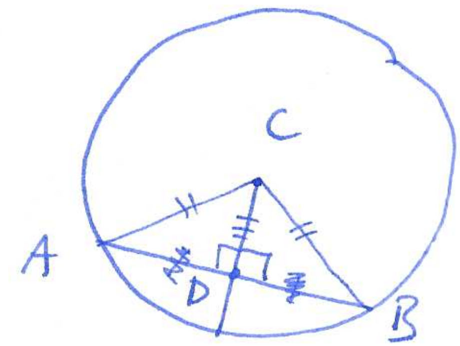
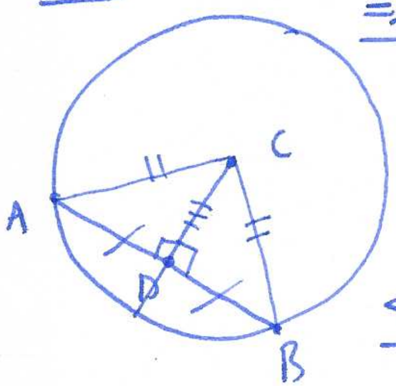
MTH 623 (4.1.3) (Book 3 Prop 3).

In a circle a radius bisects a chord not through the center iff the radius and the chord are $\perp$



Proof. Lets suppose D is the midpoint of AB.



$\Rightarrow$ construct DC

$\triangle ACD \cong \triangle BCD$ congruent (SSS)

$\angle ADC = \angle BDC \Rightarrow$ both angles 90°

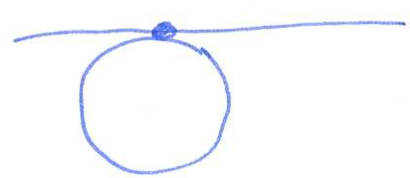
$\Leftarrow$ assume D lies on AB, and $CD \perp AB$.

~~two triangles are congruent~~

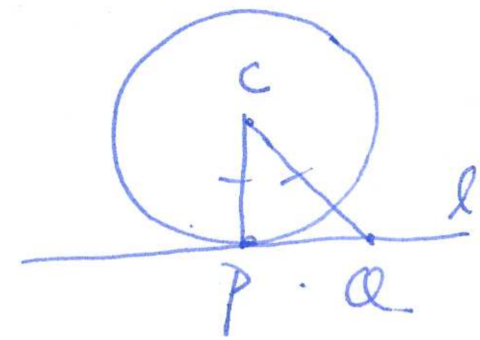
Pythagoras thm $\Rightarrow \triangle ACD \cong \triangle BCD$.

$\Rightarrow |AD| = |BD| \Rightarrow$ radius bisects AB $\square$.

Defn A straight line is tangent to a circle if it intersects it in exactly one point.



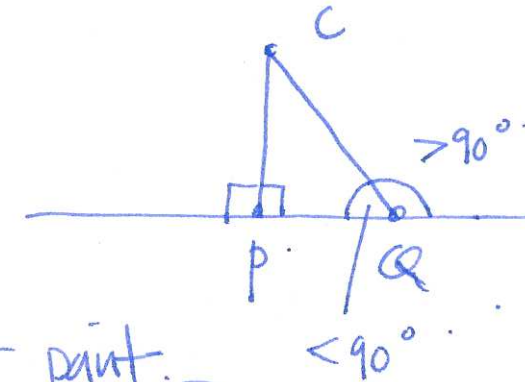
(4.1.4) (B3 P16,18) If a straight line intersects a circle, it is tangent iff it is $\perp$ to the radius at the point of intersection.



__Proof__ ($\Rightarrow$) assume l tangent:

__claim__: P is the closest point on l to C

__Proof__ (of claim) follows from the fact that greater angles are opposite greater sides in triangles. $\angle$ at P $>$ $\angle$ at Q.



$\Rightarrow |CQ| > |CP|$ so P is closest point. $\square$ claim.

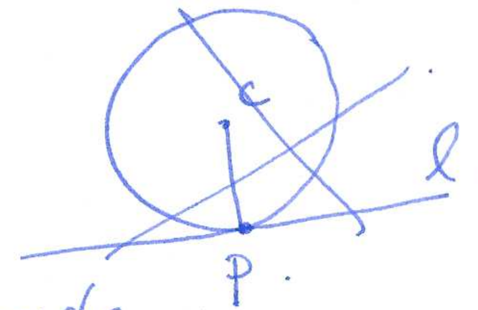
so any other point Q on l is now further from C

so Q lies outside the circle. $\square$ ($\Rightarrow$).

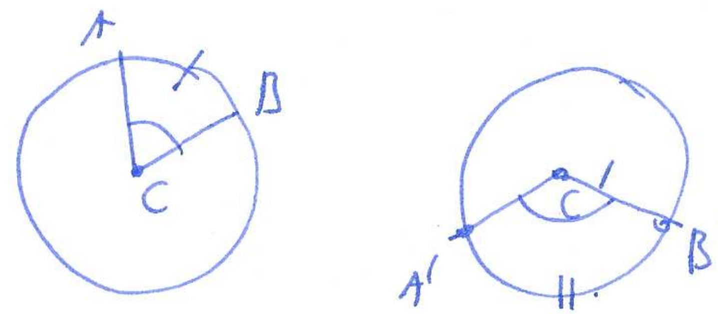
$\Leftarrow$ ~~assume ℓ is tangent to the circle.~~

assume ℓ is $\perp$ to radius at P.

this implies P is the closest point on ℓ to C. $\Rightarrow$ all other points on ℓ lie outside circle $\Rightarrow$ ℓ intersects circle at P only $\Rightarrow$ tangent. $\square$

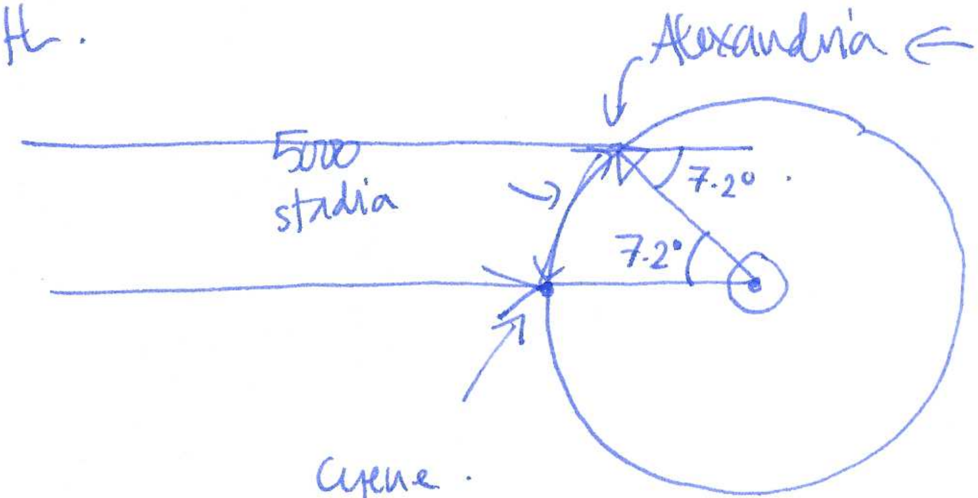


(4.1.5) B6 P33 In equal circles, central angles are proportional to the arcs they subtend.



Proof only works for rational ratios! $\square$

Erasthanes ~ 200BC estimated the circumference of the earth.



Alexandria ← at midday on same day sun makes an angle of 7.2°. at a different spot.

Cyene ↑ at midday on equinox sun shines vertically down a 50ft well.

1 stadia ~ 187m

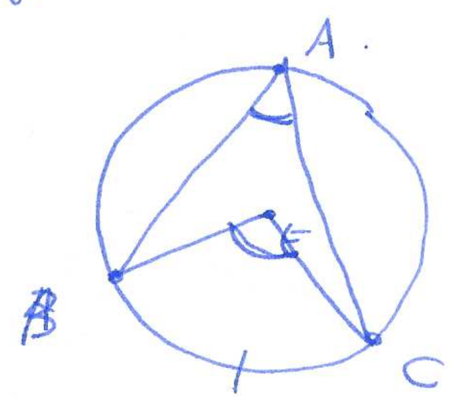
$$\frac{\text{circumference of the earth}}{5000} = \frac{360°}{7.2°} \approx 250,000 \text{ stadia}.$$

$$\approx 45,000 \text{ km}.$$

actual answer ≈ 40,000 km   accurate to ≈ 12%.

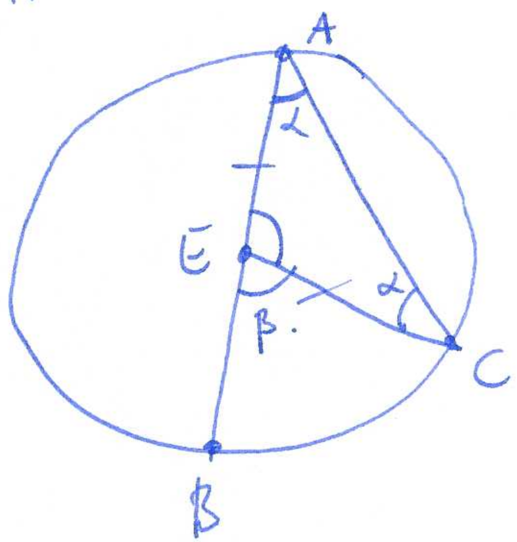
§4.2 The non-neutral geometry (Euclidean) of the circle.

(4.2.1) B3 P20 The angle subtended at the center of a circle is twice the angle subtended at the circumference. $\angle BEC = 2\angle BAC$



Proof consider the following special case:

suppose E lies on AB.



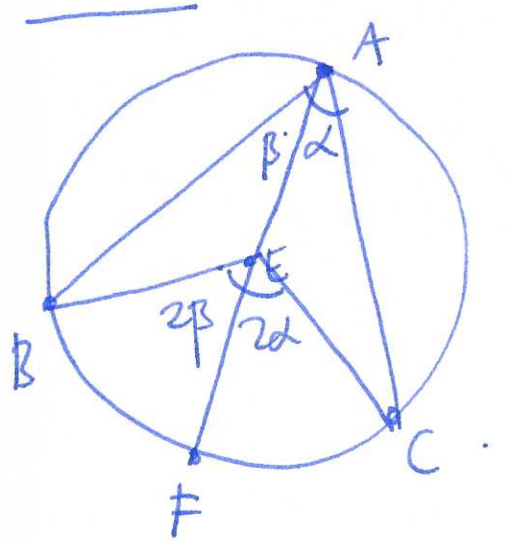
$|AE| = |EC| \Rightarrow \Delta ACE$ is iscoceles.

so $\angle AEC = 180^\circ - 2\alpha$

$\beta = 180 - (180 - 2\alpha) = 2\alpha$.

$\angle BEC = 2\angle BAC$. $\square$ (special case).

<u>case 2</u>. E lies in the interior of $\triangle ABC$.



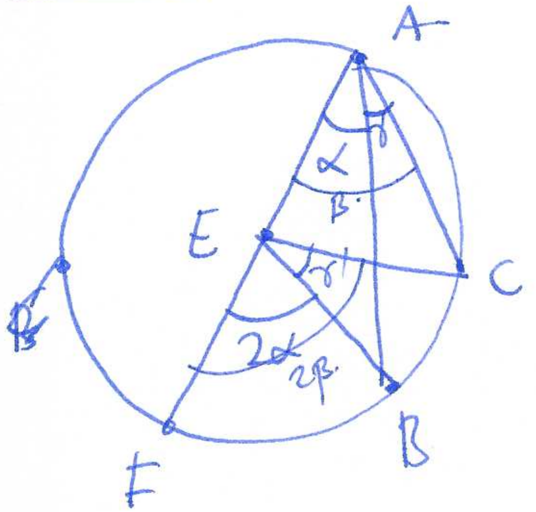
extend AE to be a diameter AF.

by previous special case

$\angle BAC = \alpha + \beta$

$\angle BEC = 2\alpha + 2\beta = 2 \angle BAC$ $\square$ (case 2).

<u>case 3</u>. E does not lie in interior of $\triangle ABC$



extend AE to be a diameter AF

special case $\Rightarrow$ $\angle BAE = \alpha$, then $\angle BEF = 2\alpha$.

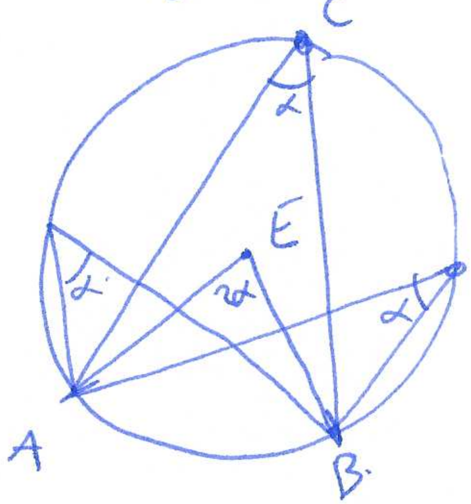
also $\Rightarrow$ $\angle CAE = \beta$, then $\angle FEC = 2\beta$.

$\angle BAC = \gamma = \beta - \alpha$

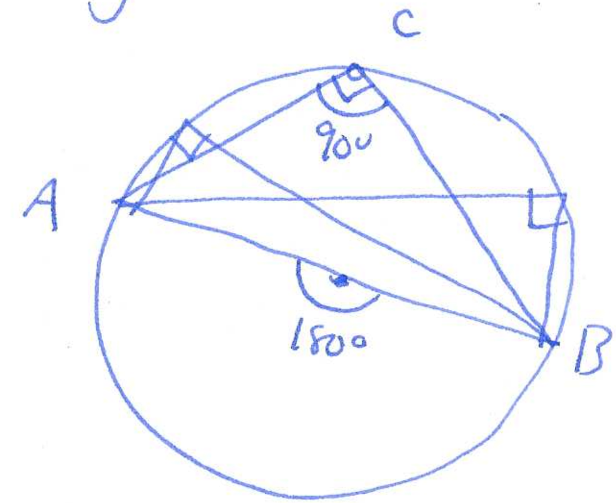
$\angle BEC = \gamma' = 2\beta - 2\alpha = 2\gamma$ $\square$ (case 3).

$\square$

Corollary (4.2.2) B3 P21 For a circle, for a given arc all angles subtended at the circumference are equal. $\square$.



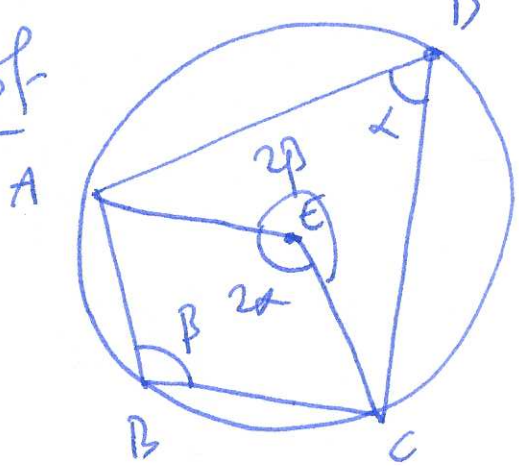
Corollary B3 P31 In a circle the angle subtended by a diameter is 90°. $\square$.



Defn A polygon is cyclic if all of its vertices lie on a circle.

(4.2.6) B3 P22 The opposite angles of a cyclic quadrilateral add up to 180°.

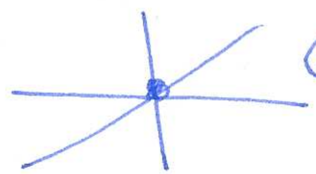
Proof

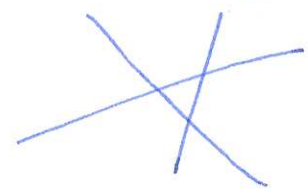


Let ABCD be four vertices of a cyclic quadrilateral lying on a circle with center E

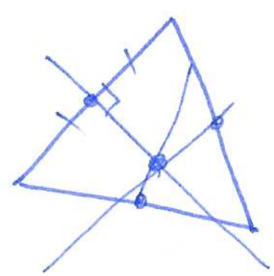
$$2\alpha + 2\beta = 360^\circ \Rightarrow \alpha + \beta = 180^\circ. \quad \square$$

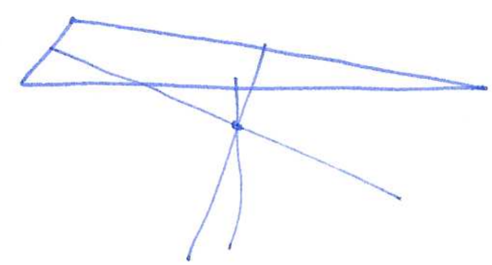
Defn 3 lines are concurrent if they meet at a common point.

 ← concurrent

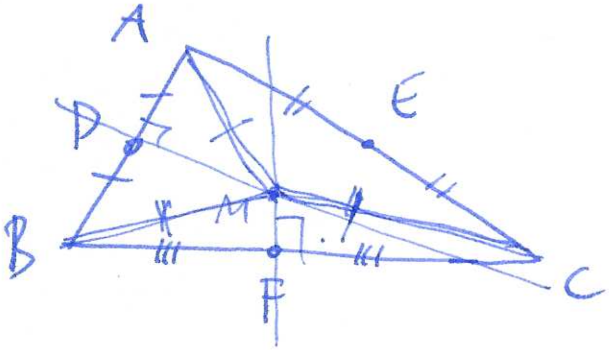
 ← not concurrent

(4.2.7) The perpendicular bisectors of the sides of a triangle are concurrent.





Proof



Let D be the midpoint of AB
E AC
F BC

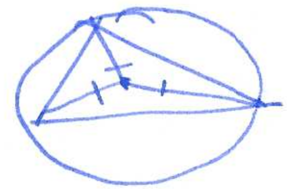
construct ⊥ lines at D and F, and let them intersect at M.

recall: ⊥ bisector to AB = set of equidistant points to A and B.

So $|AM| = |BM|$, also $|BM| = |CM|$.

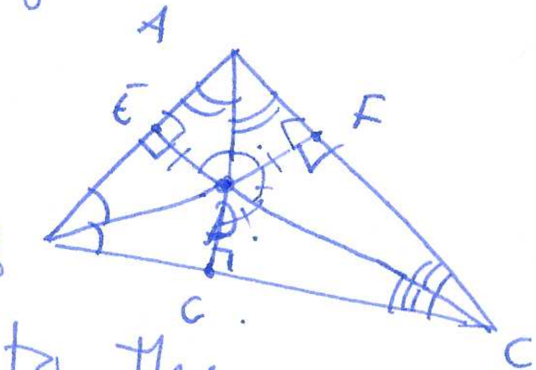
So $|AM| = |CM| \Rightarrow M$ lies on ⊥ bisector to AC. □.

(4.2.8) B4 P5 (Corollary) can circumscribe a circle about a given triangle. □



all triangles are cyclic.

(4.2.9) ~~(B4 P5)~~ The angle bisectors of a triangle are concurrent.



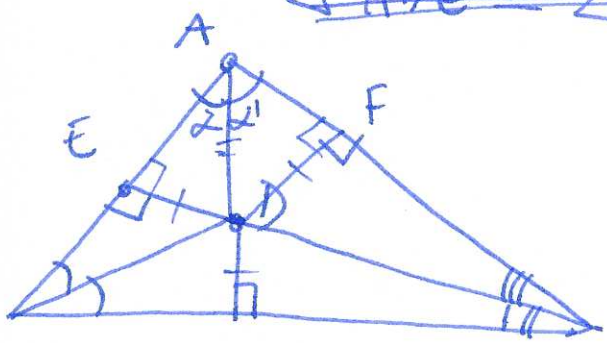
Proof Let the angle bisectors for ~~B~~ B and C meet at D. Construct the perpendiculars to the sides through D.

claim $\triangle CDG \cong \triangle CDF$ (1), because AAA, and the ratio of sides is 1

similarly $\triangle BDG \cong \triangle BDE$ (2)

finally ~~$\triangle ADE \cong \triangle ADF$, again AAA~~

~~$\triangle ADE \cong \triangle ADF$, again by AAA and ratio of sides is 1~~.



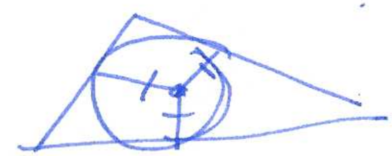
(1) $|DG| = |DF|$

(2) $|DG| = |DE|$

Pythagoras $\Rightarrow |AE| = |AF|$.

$\Rightarrow \triangle AED \cong \triangle ADF$ by (SSS) so $\alpha = \alpha'$. $\square$

(4.2.10) B4 P4 Can inscribe a circle inside a given triangle.



Proof corollary of above $\square$.