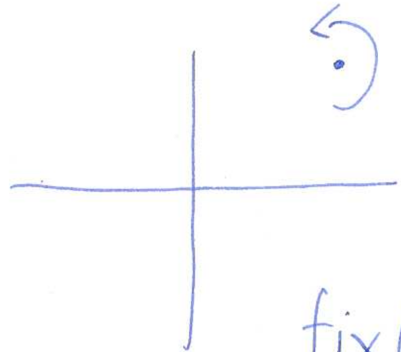
$g \in \mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$..

$h \in \mathrm{Isom}(\mathbb{R}^2)$



rotation

~~fix~~

$\mathrm{fix}(g) = \{ z \in \mathbb{C} \mid g(z) = z \}$.

hgh^{-1}

$w \in \mathrm{fix}(g)$.

<u>claim</u> $h(w) \in \mathrm{fix}(hgh^{-1})$

$$hgh^{-1}h(w) = h(g(w)) = h(w).$$

in fact $\mathrm{fix}(hgh^{-1}) = h\,\mathrm{fix}(g)$.

Möbius maps $\circlearrowright$ $\mathbb{C} \cup \{\infty\}$.

up to conjugacy:

rotations / elliptic $\quad z \mapsto e^{i\theta} z \quad \theta \in (0, 2\pi)$

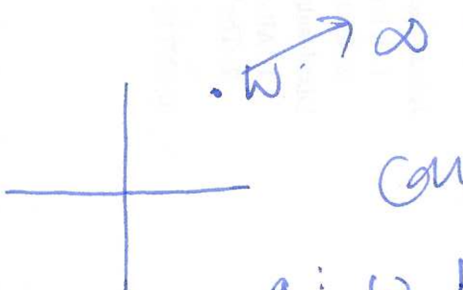
parabolics. $\quad z \mapsto z+1$

translations / hyperbolic / loxodromics $\quad z \mapsto \lambda z \quad \lambda \in \mathbb{R}_{>0}$ $\neq 0$.

<u>observation</u> every Möbius map has a fixed point.

$$f(z) = \frac{az+b}{cz+d} = z \qquad az+b = cz^2+dz.$$

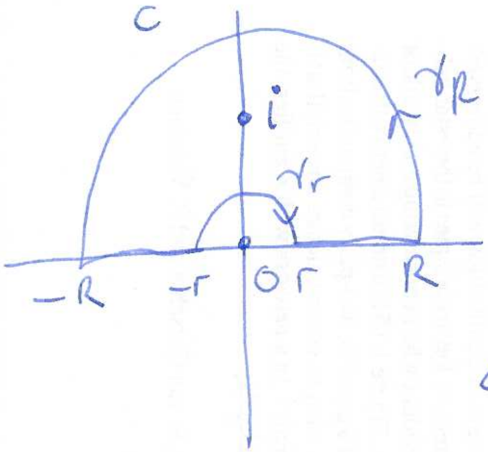
$$cz^2 + (d-a)z - b = 0$$



conjugate by $\quad g f g^{-1}$ fixes ∞.

$g : w \longmapsto \infty \qquad : z \longmapsto az+b.$

real integrals. $\displaystyle\int_0^\infty \frac{\ln(x)}{(x^2+1)^2}\,dx.$ $\qquad f(z) = \dfrac{\ln(z)}{(z^2+1)^2}.$



choose branch of $\ln(z)$

$-\pi < \operatorname{Im} \ln(z) = \arg z \leq \pi$

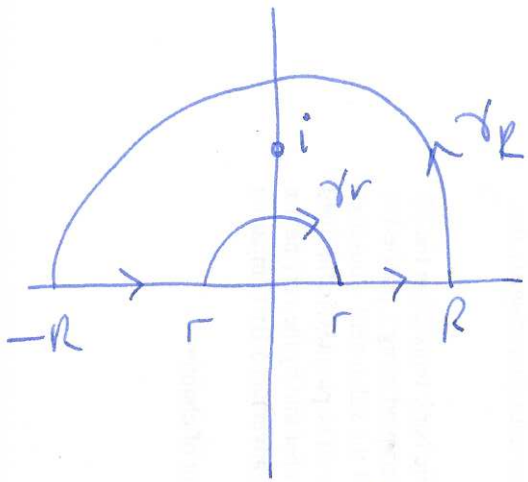
$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_i f(z)$$

$$\boxed{\ln(i) = \frac{\pi}{2} i}$$

$$\operatorname*{Res}_i f(z) = \lim_{z\to i} \frac{d}{dz} \frac{(z-i)^2 \ln(z)}{(z^2+1)^2} = \lim_{z\to i} \frac{d}{dz} \frac{\ln(z)}{(z+i)^2}.$$

$$= \lim_{z\to i} \frac{(z+i)^2 \frac{1}{z} - 2(z+i)\ln(z)}{(z+i)^4} = \frac{-4\frac{1}{i} - 4i\frac{\pi}{2}i}{16}.$$

$$= \frac{4i + 2\pi}{16} = \frac{2i+\pi}{8} = \frac{\pi + 2i}{8}$$



$$\int_C f(z)\,dz = \frac{\pi+2i}{8}\,2\pi i = -\frac{\pi}{2} + \frac{\pi^2}{4} i$$

$$\int_{-R}^{r} + \underbrace{\int_{\gamma_r}}_{2} + \int_{r}^{R} + \underbrace{\int_{\gamma_R}}_{1}$$

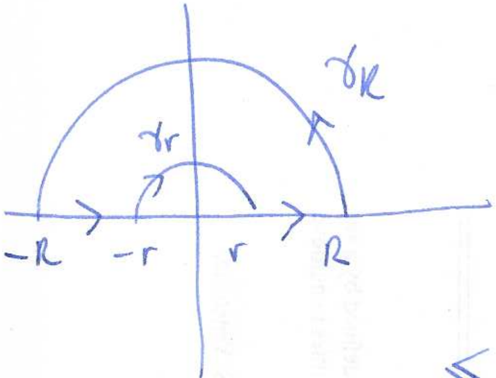
1 γ_R $\quad z = Re^{i\theta} \quad 0 \le \theta \le \pi$

$$\left|\int_{\gamma_R} \frac{\ln(z)}{(z^2+1)^2}\,dz\right| \le \pi R \max\left|\frac{\ln(z)}{(z^2+1)^2}\right| \quad \circledast$$

$$|\ln z| = |\ln R + i\theta| \le \sqrt{(\ln R)^2 + \pi^2} \le 2\ln R$$

for $R >> 0$

$$\circledast \le \pi R\,\frac{2\ln(R)}{(R^2-1)^2} \longrightarrow 0 \text{ as } R \to \infty.$$



γ_r $\qquad \left| \int_{\gamma_r} \frac{\ln(z)}{(z^2+1)^2}\, dz \right| \qquad z(\theta) = re^{i\theta}.$

$$\leq \pi r \frac{|2\ln(1/r)|}{(1-r^2)^2}.$$

$\ln(r) = -\ln(1/r).$

$z = x + iy$

$\longrightarrow 0$ as $r \to \infty$. $\qquad |\ln(r)| = |\ln(1/r)|.$

$$\underbrace{\int_{-\infty}^{0} \frac{\ln(x)}{(x^2+1)^2}\, dx} + \int_0^\infty \frac{\ln(x)}{(x^2+1)^2} = -\frac{\pi}{2} + \frac{\pi^2}{4} i.$$

$$\underbrace{2\int_0^\infty \frac{\ln(x)}{(x^2+1)^2}\, dx} + \underbrace{\int_0^\infty \frac{\pi i}{(x^2+1)^2}\, dx} = -\underbrace{\frac{\pi}{2}} + \underbrace{\frac{\pi^2}{4} i}.$$

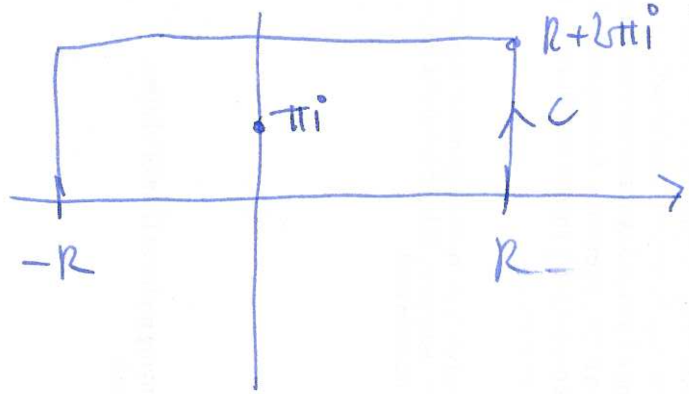
real

Example

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx \qquad (0<a<1).$$

$\mathbb{C}$.

$$f(z) = \frac{e^{az}}{1+e^{z}}$$



$$e^{z} = -1.$$

$$\pi i + 2\pi n i \qquad n\in\mathbb{Z}.$$

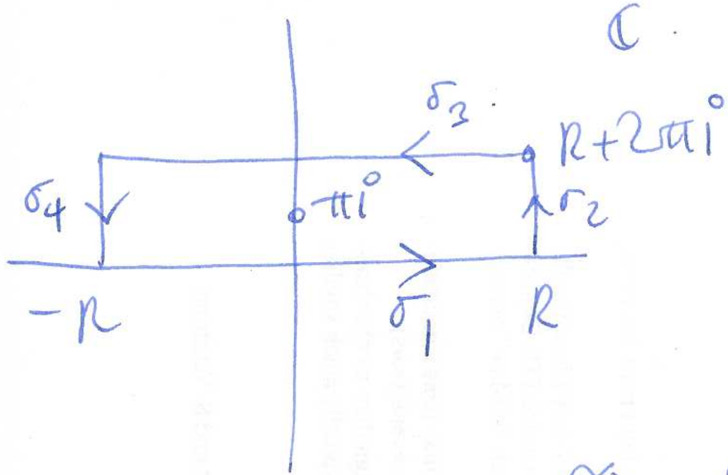
$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{\pi i} f(z)$$

$$ze^{az} - \pi i e^{az}$$

l'H.

$$\operatorname*{Res}_{\pi i} f(z) = \lim_{z\to\pi i} \frac{(z-\pi i)e^{az}}{1+e^{z}} = \lim_{z\to \pi i} \frac{e^{az} + aze^{az} - a\pi i e^{az}}{e^{z}}.$$

$$= \frac{e^{\pi i a} + \cancel{a\pi i e^{a\pi i}} - \cancel{a\pi i e^{a\pi i}}}{-1} = -e^{a\pi i}.$$

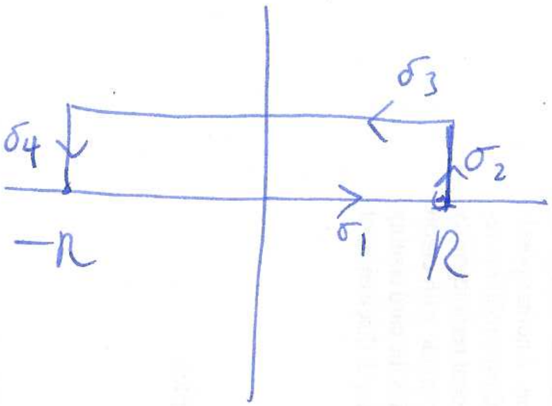


$$f(z) = \frac{e^{az}}{1+e^{z}}.$$

$$\int_C f(z)\,dz = 2\pi i \cdot -e^{a\pi i}$$
$$= -2\pi i e^{+a\pi i}$$

$$\int_{\sigma_1} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx$$

$$\int_{\sigma_3} f(z)\,dz = -\int_{-\infty}^{\infty} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx = -e^{a2\pi i}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx.$$



$$f(z) = \frac{e^{az}}{1+e^{z}}$$

$$0 < y \le 2\pi$$

$$\sigma_2 : \quad |f(z)| = \left| \frac{e^{a(R+iy)}}{1+e^{R+iy}} \right| \le \frac{e^{aR}}{e^{R}-1} \longrightarrow 0 \text{ as } R \to \infty.$$

$$\sigma_4 : \quad |f(z)| = \left| \frac{e^{a(-R+iy)}}{1+e^{-R+iy}} \right| \le \frac{e^{-aR}}{1-e^{-R}} \longrightarrow 0 \text{ as } R \to \infty.$$

$$\left(1 - e^{a2\pi i}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = -2\pi i\, e^{a\pi i}.$$

(9)

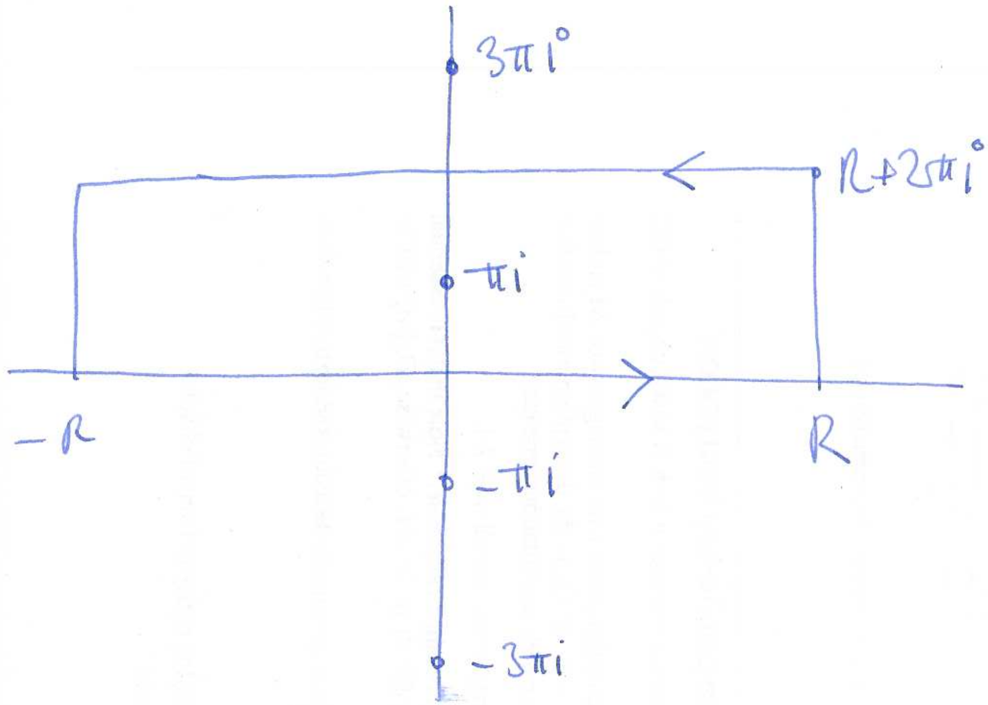
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{-2\pi i\, e^{a\pi i}}{1-e^{a2\pi i}}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$= \frac{-2\pi i}{e^{-a\pi i}-e^{a\pi i}} = \frac{\pi}{\sin(a\pi)}.$$

$$\frac{e^{x}}{1-e^{2x}} = \frac{1}{e^{-x}-e^{x}}.$$

$$\frac{e^{a\pi i}}{1-e^{a2\pi i}} \times \frac{e^{-a\pi i}}{e^{-a\pi i}} = \frac{1}{e^{-a\pi i}-e^{+a\pi i}}.$$

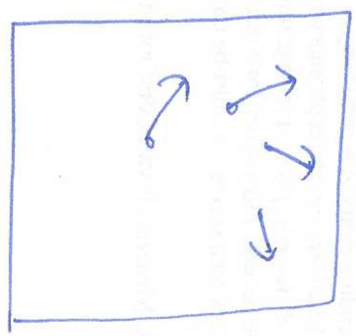


$$f(z) = \frac{e^{az}}{1+e^{z}}.$$

$$e^{R+2\pi i} = e^{R}.$$

§ 15.1 Fluid dynamics

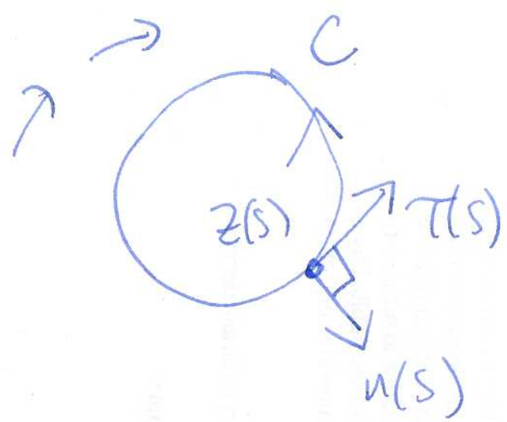
(2d case) $\leftarrow$ (3d case where the flow is planar).



at each point the fluid is moving with same velocity, i.e. vector valued function $w : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$w : \mathbb{C} \longrightarrow \mathbb{C}$.

$$w(x,y) = u(x,y) + iv(x,y).$$



C has parameterization $z(s)$

<u>unit speed</u>. C has length ℓ.

$\tau(s)$ unit tangent vector to C

$n(s)$ unit normal vector to C

$$\tau(s) = z'(s) = \frac{dx}{ds} + i\frac{dy}{ds}.$$

$$n(s) = \frac{1}{i} z'(s) = +\frac{dy}{ds} - i\frac{dx}{ds}$$

$$= -i.$$

we can ~~wit~~ write the flow vector $w(s)$ in terms of τ and n.

$$w(s) = \underbrace{w_\tau \tau(s)}_{\text{tangential part}} + \underbrace{w_n n(s)}_{\text{normal part}}$$

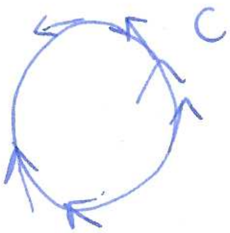
w_τ = scalar product $w(s) \cdot w_\tau(s)$

w_n = scalar product $w(s) \cdot w_n(s)$

$$w_\tau = u\frac{dx}{ds} + v\frac{dy}{ds}.$$

$$w_n = -v\frac{dx}{ds} + u\frac{dy}{ds}.$$

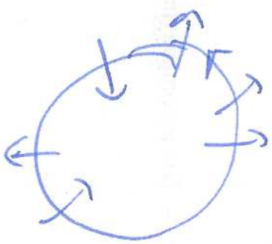
<u>assume</u> $w = u + iv$ continuous, differentiable, partial derivatives exist and are cts etc...



$$\int_C w_\tau \, ds = \int_C (u+iv)_\tau \, ds = \int_C u\,dx + v\,dy \quad (1)$$

↑ called the __circulation__ around C.
along

$$\int_C w_n \, ds = \int_C (u+iv)_n \, ds = \int_C -v\,dx + u\,dy. \quad (2)$$



↑ called the __flux__ through C.

if (1) = 0 for all closed curves C then flow is __irrotational__

if (2) = 0 for all closed curves C say flow is __solenoidal__ consider $\displaystyle\int_C \bar{w}\,dz = \int_C \overline{u+iv}\,(dx+i\,dy)$

$$= \int_C u\,dx + v\,dy + i\left(\int_C -v\,dx + u\,dy\right).$$

1 $\displaystyle\int_C u\,dx + v\,dy = \operatorname{Re}\int_C \bar{w}\,dz$.

2 $\displaystyle\int_C -v\,dx + u\,dy = \operatorname{Im}\int_C \bar{w}\,dz$

$f(z)$ is called the complex potential

<u>Thm</u> A cts diff flow $u+iv$ defined in a simply connected domain G is irrotational + solenoidal iff $u+iv = \overline{f'(z)}$ where $f(z)$ is analytic in G

<u>Proof</u> suppose $\int_C = 0$ ① ② for all closed curves C.

then $u\,dx + v\,dy$ and $-v\,dx + u\,dy$ are exact differentials

so there are real valued functions $\phi(x,y)$, $\psi(x,y)$

s.t.

$$u\,dx + v\,dy = d\phi \qquad -v\,dx + u\,dy = d\psi.$$

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \qquad -v = \frac{\partial \psi}{\partial x} \quad u = \frac{\partial \psi}{\partial y}$$

so ϕ, ψ satisfy CR equations

$$f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = u - iv.$$

$f(z) = \phi + i\psi$ analytic in C. $\overline{f'(z)} = u + iv.$ $\square$

- $u, -v$ are conjugate harmonic functions.

① $\int_C w_\tau \, ds = \int_C u\,dx + v\,dy = \mathrm{Re} \int_C f'(z)\,dz$

② $\int_C w_n \, ds = \int_C -v\,dx + u\,dy = \mathrm{Im} \int_C f'(z)\,dz$

ϕ velocity potential

ψ stream function (flow)

$\phi(x,y) = \text{const}$

equipotentials

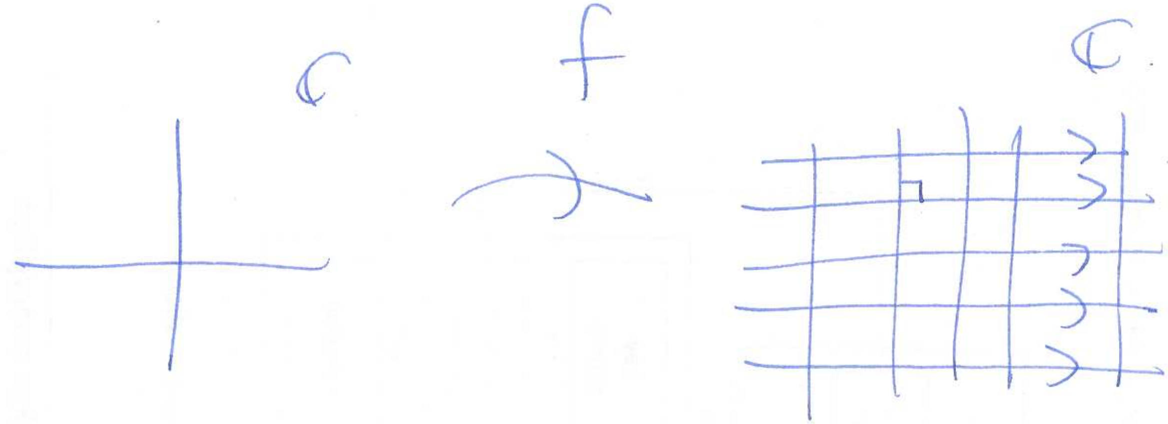
$\psi(x,y) = \text{const}$

stream lines

flow lines

the map $f(z)$ is conformal, except where $f'(z)=0$

stagnation point



$f(z)$ takes

$$\phi(x,y) = \text{const} \longmapsto \mathrm{Re}(z) = \text{const}$$

$$\psi(x,y) = \text{const} \longmapsto \mathrm{Im}(z) = \text{const}$$

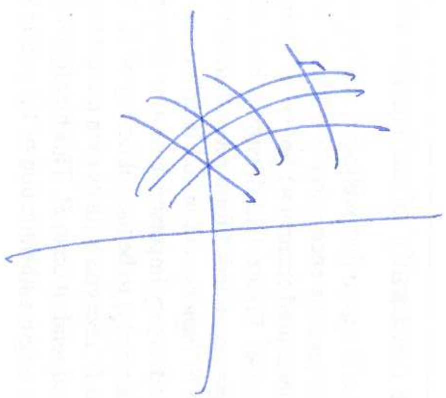
check $\phi(x,y) = c$

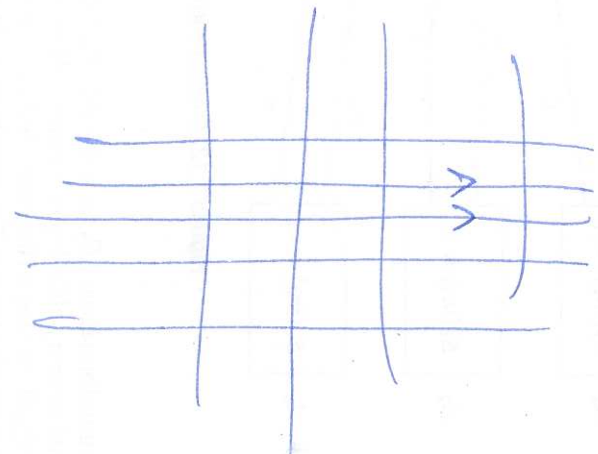
$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \longleftrightarrow u\,dx + v\,dy = 0.$$

~~velocity lines are~~

velocity is tangential to the streams lines
flow lines.

these are the actual trajectories of the flow.



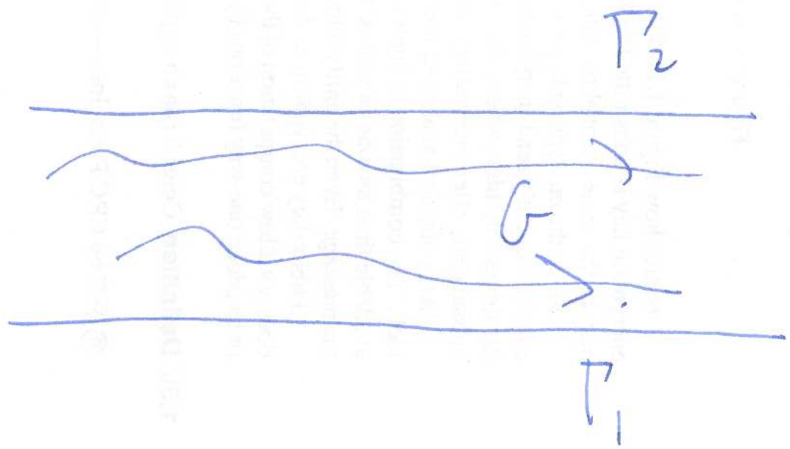


<u>Fact</u> domain G, $\partial G = \Gamma \leftarrow$ some curves in $\mathbb{C}$.

each curve in ∂G must be a flow line
stream
for the flow (the flow can't cross the boundary).

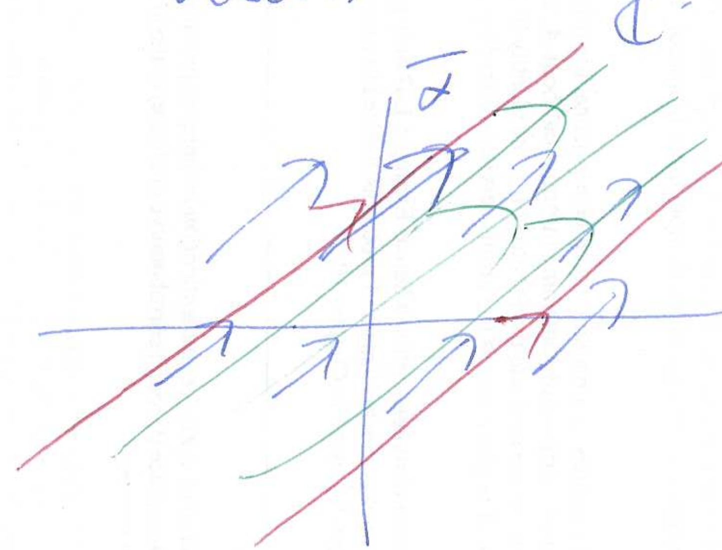
i.e. $\psi(x,y) = \text{const}$ on arcs of ∂G.

$\overset{\parallel}{\operatorname{Im}(f(z))}$.



Examples ① $f(z) = \alpha z$ ← flow on the whole plane with uniform velocity

$$\overline{f'(z)} = \overline{\alpha}$$



if $\alpha = a+ib$

$$\phi(x,y) = ax - by$$

$$\psi(x,y) = bx + ay.$$

- this also models the uniform flow in a strip whose sides are two parallel lines in direction $\overline{\alpha}$.

2 $f(z) = z^2$ flow on the whole of $\mathbb{C}$.

$\overline{f'(z)} = 2\bar{z}$ (non-uniform velocity)

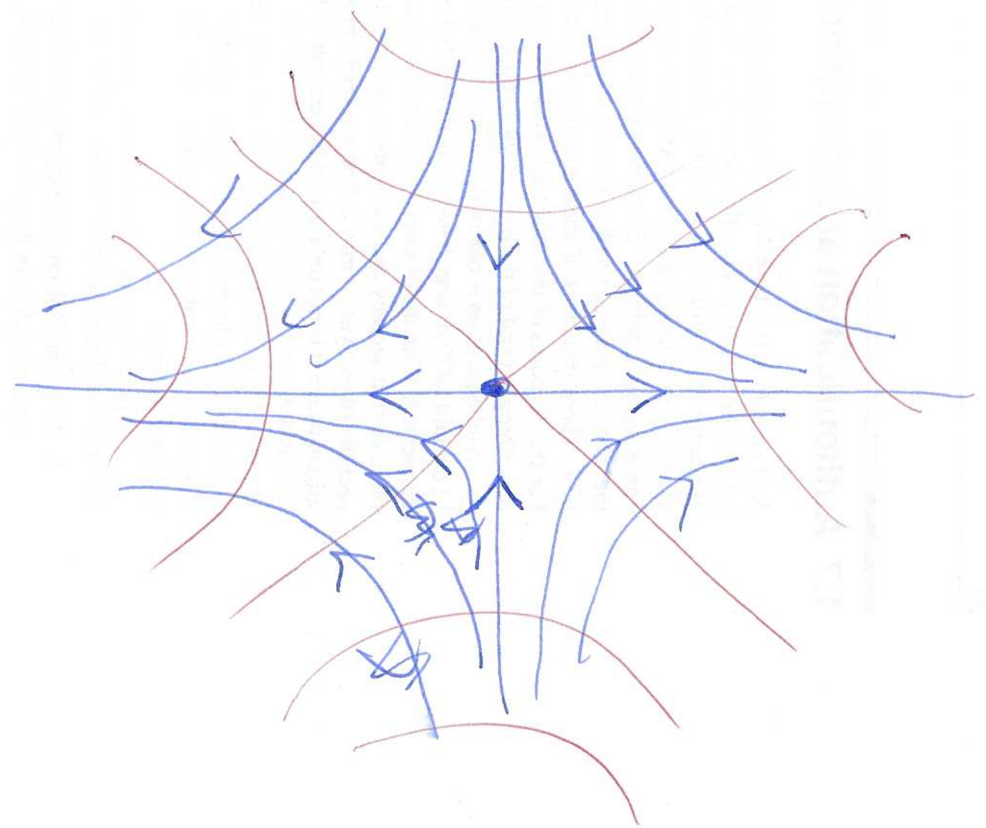
$z = x+iy$ $z^2 = x^2-y^2+i2xy$

$\phi(x,y) = x^2-y^2 = c$ equipotentials

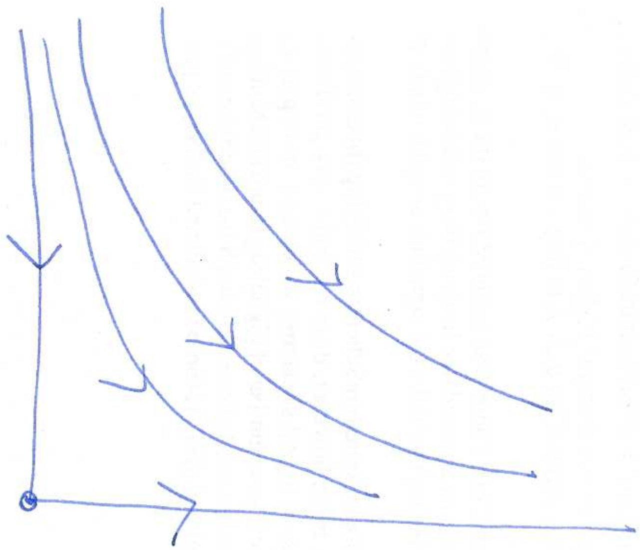
$\psi(x,y) = 2xy = c$ flow lines

$z=0$ is a stagnation point

$y = \frac{c}{2x}$



x, y axes are flow lines

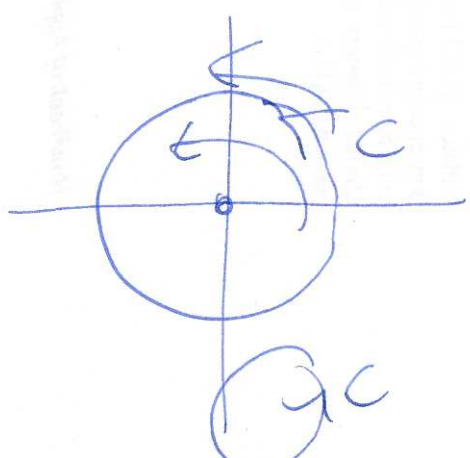


this models fluid flow around a corner.

③ circular flows

$\mathbb{C} \setminus \{0\} = G$

$k \in \mathbb{R}$

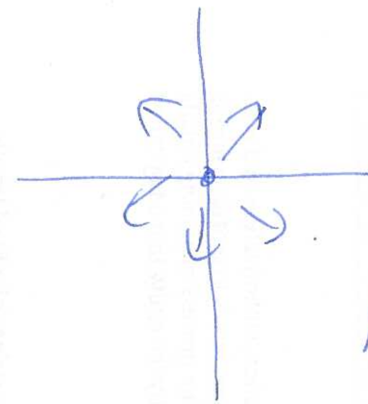


$$f(z) = \frac{k}{2\pi \cdot i} \ln(z)$$

$$\int_C w_\tau \, ds = k \quad \text{circulation}$$

$$\int_C w_n \, ds = 0$$

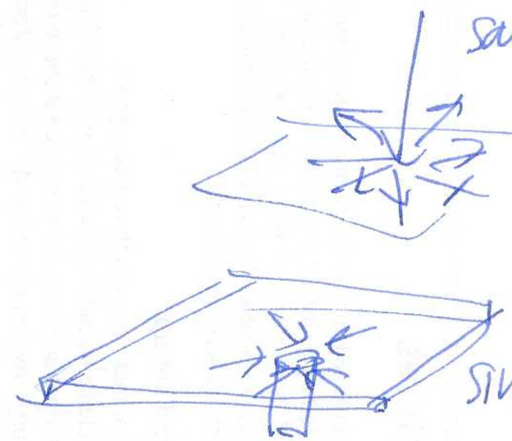
$$f(z) = \frac{\mu}{2\pi} \ln(z) . \qquad \mu \in \mathbb{R}$$



$\int \omega_T \, ds = 0$ circulation $= 0$

$\int \omega_n \, ds = \mu$ flux $= \mu$.

$\mu > 0$



$\mu < 0$

(5) flow past a cylinder