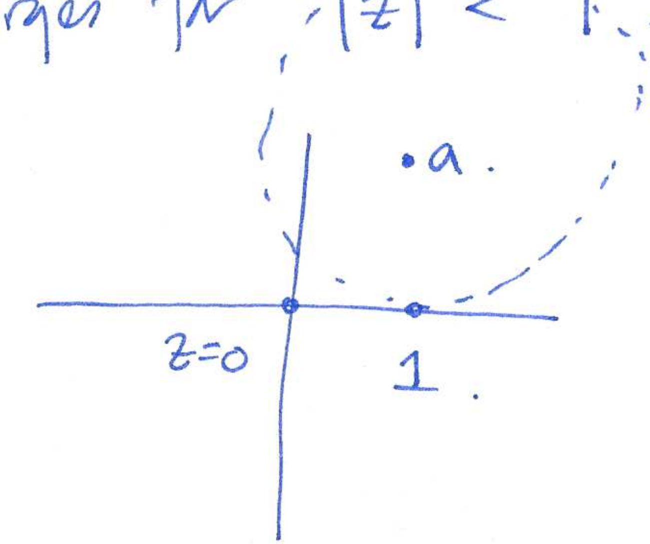
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$ converges for $|z| < 1$.



expand around a:

$$c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

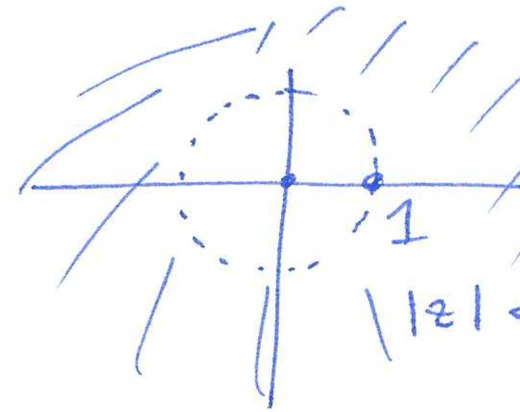
$$\frac{1}{1-a-(z-a)} \qquad \frac{1}{1-a}\;\frac{1}{1-\frac{z-a}{1-a}}$$

$$\frac{1}{1-a}\left(1 + \frac{z-a}{1-a} + \left(\frac{z-a}{1-a}\right)^2 + \left(\frac{z-a}{1-a}\right)^3 + \cdots\right)$$

converges for $\left|\frac{z-a}{1-a}\right| < 1$ $\qquad |z-a| < |1-a|$

②

$$\frac{1}{1-z} = 1+z+z^2+\cdots$$



$$\frac{1}{z}\,\frac{1}{\frac{1}{z}-1} = -\frac{1}{z}\,\frac{1}{1-1/z}.$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right).$$

$$L(z) = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\cdots$$

$\left|\frac{1}{z}\right|<1$

$|z|>1$.

$1<|z|<\infty$

Q11. 4 a)

$$\frac{z}{(z+2)^2}$$ ← Laurent series centered at $z_0=-2$.

pole of order 2.

$$\sum_{n\in\mathbb{Z}} c_n (z+2)^n.$$

$$\frac{c_{-2}}{(z+2)^2}+\frac{c_{-1}}{z+2}+c_0+c_1(z+2)+c_2(z+2)^2+\cdots$$

partial fractions:

$$\frac{z}{(z+2)^2}=\frac{A}{(z+2)^2}+\frac{B}{z+2}$$

$$=\frac{A+B(z+2)}{(z+2)^2}.$$

$z = Bz + A+2$

$B=1 \quad A=-2.$

$$=\frac{-2}{(z+2)^2}+\frac{1}{z+2}$$

b) $\dfrac{e^z+1}{e^z-1}$ at $z_0 = 2\pi i$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\frac{1+z+\frac{z^2}{2!}+\cdots+1}{1+z+\frac{z^2}{2!}+\cdots-1} = \frac{2+z+\frac{z^2}{2!}+\cdots}{z+\frac{z^2}{2!}+\cdots} \leftarrow$$

centered at $z=0$!

want Laurent series $\sum_{n\in\mathbb{Z}} c_n (z-2\pi i)^n$

b) $\dfrac{e^z+1}{e^z-1}$ centered $z_0 = 2\pi i$ $\displaystyle\sum_{n\in\mathbb{Z}} c_n (z-2\pi i)^n$.

<u>claim</u> : simple pole at $z = 2\pi i$.

$$\frac{e^z+1}{e^z-1} = \frac{c_{-1}}{z-2\pi i} + c_0 + c_1(z-2\pi i) + c_2(z-2\pi i)^2 + \cdots$$

$$\frac{(z-2\pi i)(e^z+1)}{(e^z-1)} = c_{-1} + c_0(z-2\pi i) + c_1(z-2\pi i)^2 + \cdots$$

$$c_{-1} = \lim_{z\to 2\pi i} \frac{(z-2\pi i)(e^z+1)}{(e^z-1)}$$

$$c_{-1} = \lim_{z \to 2\pi i} \frac{(z-2\pi i)(e^z+1)}{(e^z-1)} = 2 \lim_{z\to 2\pi i} \frac{z-2\pi i}{e^z-1}.$$

1 l'Hôpital : $$2 \lim_{z \to 2\pi i} \frac{1}{e^z} = 2. = c_{-1}.$$

$$\underbrace{\frac{2}{z-2\pi i}} + c_0 + c_1(z-2\pi i) + \cdots$$

2 $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$2 \lim_{z\to 2\pi i} \frac{z-2\pi i}{e^z-1}. = 2 \lim_{w \to 0} \frac{w}{e^{w+2\pi i}-1}$$

$w = z-2\pi i$

$$= 2 \lim_{w\to 0} \frac{w}{e^w \cdot \underbrace{e^{2\pi i}}_{1} - 1}$$

$$c_{-1} = 2 \lim_{w \to 0} \frac{w}{e^w - 1}$$

$$= 2 \lim_{w \to 0} \frac{w}{1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots - 1}$$

$$= 2 \lim_{w \to 0} \frac{w}{w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots}$$

$$= 2 \lim_{w \to 0} \frac{1}{1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots} = 2.$$

c) $\dfrac{z-1}{\sin^2 z}$ $\quad z_0 = 0.$ $\quad \sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots$

$$\sin^2 z = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right)^2.$$

pole of order 2.

$$= z^2 + b_4 z^4 + \cdots$$

$$\frac{z-1}{\sin^2 z} = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

find c_{-2}: $\quad \dfrac{z^2(z-1)}{\sin^2 z} = c_{-2} + c_{-1} z + c_0 z^2 + \cdots$

$$c_{-2} = \lim_{z \to 0} \frac{z^2(z-1)}{\sin^2 z} = -\lim_{z \to 0} \frac{z^2}{\sin^2 z}$$

$$c_{-2} = -\lim_{z\to 0} \frac{z^2}{\sin^2 z}.$$

$$c_{-2} = -1$$

① l'H: $$-\lim_{z\to 0} \frac{2z}{\underbrace{2\sin z\cos z}_{\sin 2z}} \overset{l'H}{=} -\lim_{z\to 0} \frac{2}{2\cos 2z} = -1.$$

② $$-\left(\lim_{z\to 0} \frac{z}{\sin z}\right)^2 \overset{l'H}{=} -\left(\lim_{z\to 0} \frac{1}{\cos z}\right)^2 = -1.$$

③ $$\sin z = z - \frac{z^3}{3!} + \cdots \qquad -\left(\lim_{z\to 0} \frac{z}{\sin z}\right)^2.$$

$$-\left(\lim_{z\to 0} \frac{z}{z - \frac{z^3}{3!} + \cdots}\right)^2 = -\left(\lim_{z\to 0} \frac{1}{1 - \frac{z^2}{3!} + \cdots}\right) = -1$$

④ $$-\lim_{z\to 0} \frac{z^2}{\sin^2 z} = -\lim_{z\to 0} \frac{z^2}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^2}.$$

$$- \lim_{z \to 0} \frac{z^2}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right)^2} .$$

$$- \lim_{z \to 0} \frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^5}{5!} - \cdots \right)^2}$$

$$- \lim_{z \to 0} \frac{1}{1 - \frac{2z^2}{6} + \cdots} = -1 .$$

$$\frac{z-1}{\sin^2 z} = \frac{-1}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

$$\frac{z^2(z-1)}{\sin^2 z} = -1 + c_{-1} z + c_0 z^2 + \cdots$$

$$\frac{d}{dz}\left(\frac{z^2(z-1)}{\sin^2 z}\right) = c_{-1} + 2c_0 z + \cdots$$

apply L'H. 3 times

$$\lim_{z\to 0} \frac{d}{dz} \frac{z^2(z-1)}{\sin^2 z} = c_{-1}.$$

2

$$\frac{z-1}{\sin^2 z} = \frac{-1}{z^2} + \frac{c_{-1}}{z} + c_0 + \cdots$$

$$\frac{z-1}{\sin^2 z} + \frac{1}{z^2} = \frac{c_{-1}}{z} + c_0 + c_1 z + \cdot$$

$$\lim_{z\to 0} z\left[\frac{z-1}{\sin^2 z} + \frac{1}{z^2}\right] = c_{-1}.$$ use l'H.

3 $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots$

$$\frac{z-1}{\sin^2 z} = \frac{z-1}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^2}.$$

$$\frac{z-1}{z^2} \quad \frac{1}{1-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\cdots}$$

$$\frac{1}{1-w} = 1+w+w^2+\cdots$$

$$\left(-\frac{1}{z^2}+\frac{1}{z}\right)\cdot\frac{1}{1-\left(\frac{z^3}{3!}-\frac{z^5}{5!}+\cdots\right)}.$$

$$\left(-\frac{1}{z^2}+\frac{1}{z}\right)\cdot\left(1+\left(\frac{z^3}{3!}-\frac{z^5}{5!}+\cdots\right)+\left(\frac{z^3}{3!}-\frac{z^5}{5!}\cdots\right)^2+\left(\quad\right)^3+\cdots\right.$$

$$z^3. \qquad z^6 \qquad z^9.$$

$$\frac{z^3}{6}$$

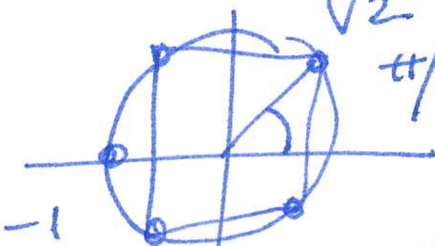
$$\left(-\frac{1}{z^2}+\frac{1}{z}\right)\cdot\left(1+\frac{z^3}{6}+\cdots\right) = \underbrace{-\frac{1}{z^2}+\frac{1}{z}}_{c_{-1}=+1}+\cdots\cdots$$

Q 11.10 a) $\dfrac{1}{z(1-z^2)} = \dfrac{1}{z(1-z)(1+z)} \leftarrow$ singularities

$0, 1, -1$

simple poles.

b) $\dfrac{z^4}{1+z^4}$ $\quad z^4 = -1$



$e^{i\frac{\pi}{4} + \frac{\pi}{2}n} \quad n = 0, 1, 2, 3.$

$\dfrac{\pm 1 \pm i}{\sqrt{2}}$ simple poles.

c) $\dfrac{z^5}{(1-z)^2} \leftarrow$ sing $z = 1$.

pole of order 2.

sing 0 simple

d) $\dfrac{1}{z(z^2+4)^2} = \dfrac{1}{z(z+2i)^2(z-2i)^2}$ $\quad \pm 2i$. order 2.

e) $\dfrac{e^z}{1+z^2}$ sing $z = \pm i$ poles of order 1.

f) $\dfrac{z^2+1}{e^z}$ sing : no sing points !

g) $z e^{-z}$ sing : no sing points !

h) $\dfrac{1}{e^z - 1} - \dfrac{1}{z}$ sing pts : 0 $\quad e^z = 1$

$2\pi i n \quad n \in \mathbb{Z}$ $\quad z = 2\pi i n$

$n \in \mathbb{Z}$

<u>check $n=0$</u> :

$$\frac{1}{e^z-1} - \frac{1}{z} \qquad \underline{z=0}.$$

1. $$\lim_{z\to 0} \frac{1}{e^z-1} - \frac{1}{z} = \lim_{z\to 0} \frac{z-e^z+1}{ze^z-z}.$$

$$\overset{l'H}{=} \lim_{z\to 0} \frac{1-e^z}{e^z+ze^z-1} \overset{l'H}{=} \lim_{z\to 0} \frac{-e^z}{2e^z+ze^z} = -\frac{1}{2} \neq \infty$$

$z=0$ $\underline{\text{regular pt}}$.

$\overline{\text{removable sing}}$

singular pts

$2\pi i n$

$n \in \mathbb{Z} \setminus \{0\}$.

(2) $\dfrac{1}{e^z - 1} - \dfrac{1}{z}$. $\qquad e^z = 1 + z + \dfrac{z^2}{2!} + \cdots$

$$\frac{1}{\left(\cancel{1} + z + \frac{z^2}{2!} + \cdots\right) - \cancel{1}} - \frac{1}{z} = \frac{1}{z + \frac{z^2}{2!} + \cdots} - \frac{1}{z}.$$

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots$$

$$\frac{1}{z}\left[\frac{1}{1 + \underbrace{\left[\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right]}_{-w.}} - 1\right].$$

$$\frac{1}{z}\left[\cancel{1} + \left(-\frac{z}{2!} - \frac{z^2}{3!} - \cdots\right) + \left(-\frac{z}{2!} - \frac{z^2}{3!} - \cdots\right)^2 + \cdots \quad - \cancel{1}.\right].$$

$$\frac{1}{z}\left[-\frac{z}{2}. + b_2 z^2 + \cdots\right] \qquad -\frac{1}{2} + b_2 z + \cdots \qquad -\frac{1}{2}$$

$1 + z + z^2 + z^3 + \cdots$ $\quad |z| < 1.$

$1 + 1 + 1 + 1 + \cdots\cdots$

$$\sum_{n \in \mathbb{N}} a_n.$$

$1 + 2 + 3 + 4 + \cdots$

$1 - 1 + 1 - 1 + \cdots\cdots$

$\nearrow 1.$

if $\sum a_n$ and $\sum b_n$ converge then

$$\sum (a_n + b_n) = \sum (a_n) + \sum (b_n).$$

$(a_n)_{n \in \mathbb{N}}.$

$$\mathbb{N}^{\mathbb{N}}. \longrightarrow \mathbb{R}$$

$$(a_n) \longmapsto \sum (a_n)\ )$$

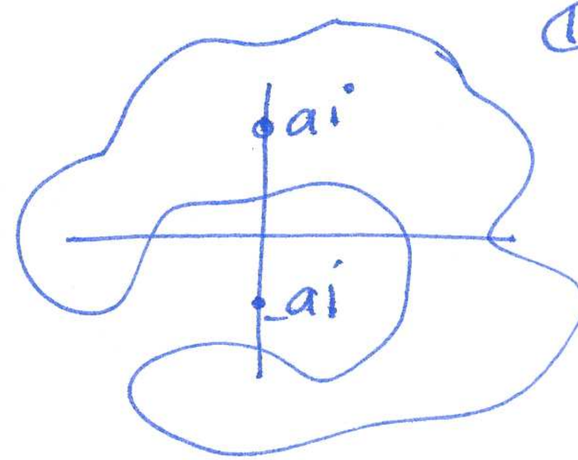
Evaluating real integrals

$a \in \mathbb{R}$.

$a > 0$

Example $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3}\,dx$

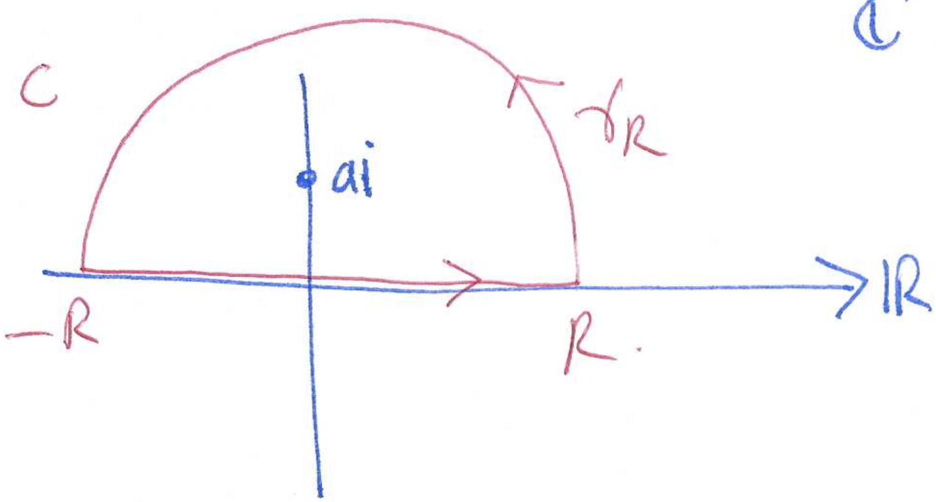
$\mathbb{C}$.



consider $f(z) = \dfrac{1}{(z^2+a^2)^3}$.

singular points : $\pm ai \leftarrow$ poles of order 3.

$\mathbb{C}$.



$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=ai} f(z)$$

$$f(z) = \frac{1}{(z^2+a^2)^3}. \qquad \operatorname*{Res}_{z=ai} f(z) \qquad \text{pole of order 3.}$$

<u>recall</u>

$$\frac{c_{-3}}{(z-ai)^3} + \frac{c_{-2}}{(z-ai)^2} + \frac{c_{-1}}{(z-ai)} + c_0 + c_1(z-ai) + \cdots$$

$$\operatorname*{Res}_{z=ai} f(z) = \cancel{\lim \frac{1}{2!}\frac{d^2}{dz^2}(z-ai)^3 f(z)}$$

$$c_{-1} = \lim_{z\to ai} \frac{1}{2!}\frac{d^2}{dz^2}(z-ai)^3 f(z).$$

$$(z+ai)^3(z-ai)^3$$

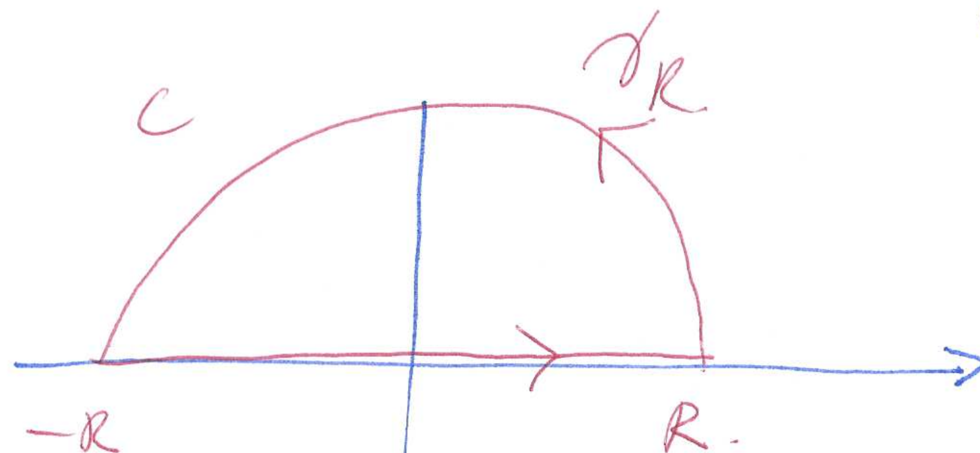
$$= \lim_{z\to ai} \frac{1}{2}\frac{d^2}{dz^2}\left[(z-ai)^3 \frac{1}{(z^2+a^2)^3}\right].$$

$$\lim_{z\to ai} \frac{1}{2}\frac{d^2}{dz^2}\frac{1}{(z+ai)^3} = \lim_{z\to ai}\frac{1}{2}\frac{d}{dz}\frac{-3}{(z+ai)^{-4}}$$

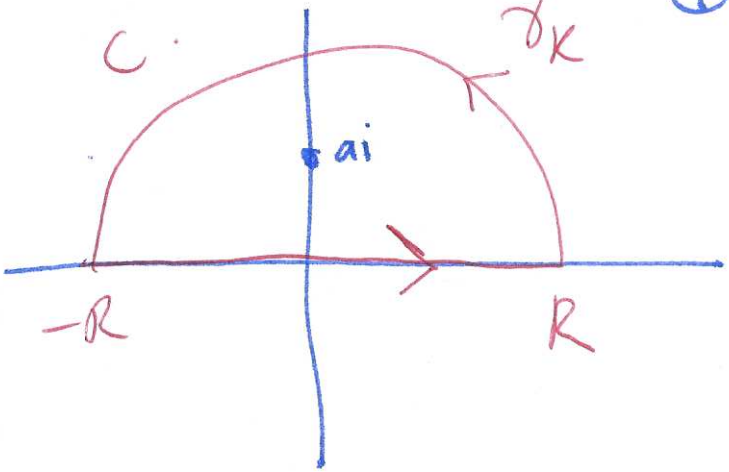
$$= \lim_{z\to ai}\frac{1}{2}\frac{12}{(z+ai)^5} = \frac{1}{2}\cdot\frac{12}{(2ai)^5} = \frac{6}{32a^5 i}$$

$\mathbb{C}$.

$$\operatorname*{Res}_{ai} f(z) = \frac{3}{16a^5 i}$$



$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{ai} f(z) = \frac{3\times 2\pi \cancel{i}}{16a^5 \cancel{i}} = \frac{3\pi}{8a^5}.$$



$\mathbb{C}$.

$$\int_C f(z)\,dz = \frac{3\pi}{8a^5}$$

$$= \underbrace{\int_{-R}^{R} f(x)\,dx}_{} + \int_{\gamma_R} f(z)\,dz = \frac{3\pi}{8a^5}.$$

$$f(z) = \frac{1}{(a^2+z^2)^3}. \qquad \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(a^2+x^2)^3}\,dx = \int_{-\infty}^{\infty} \frac{1}{(a^2+x^2)^3}\,dx.$$

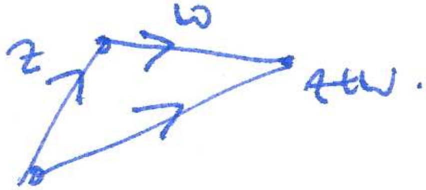
$$\left| \int_{\gamma_R} \frac{1}{(a^2+z^2)^3}\,dz \right| \le \int_{\gamma_R} \left| \frac{1}{(a^2+z^2)^3} \right| dz.$$

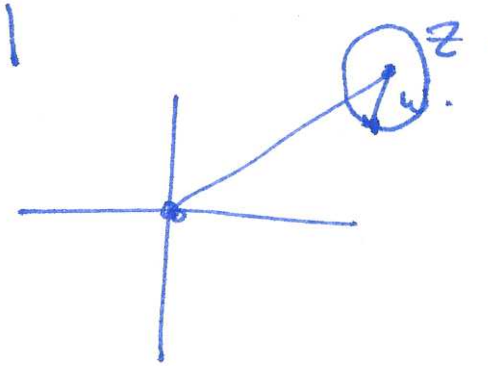
$$\frac{1}{|(a^2+z^2)^3|} = \frac{1}{|a^2+z^2|^3}.$$

$|z^2+a^2|$ on γ_R $|z| = R$.

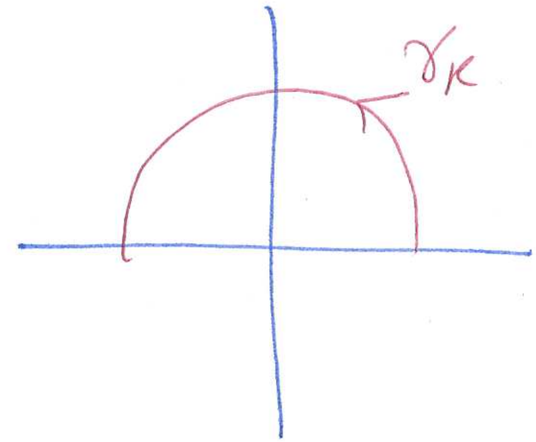
$\sim R^2 - |a|^2$

$|z+w| \le |z|+|w|$. $|z|-|w| \le |z+w|$





$$\left|\int_\gamma \frac{1}{(a^2+z^2)^3}\,dz\right| \le \int_{\gamma_R} \frac{1}{|a^2+z^2|^3}\,dz.$$



$$\frac{1}{|a^2+z^2|^3} \le \frac{1}{(R^2-|a|^2)^3}.$$

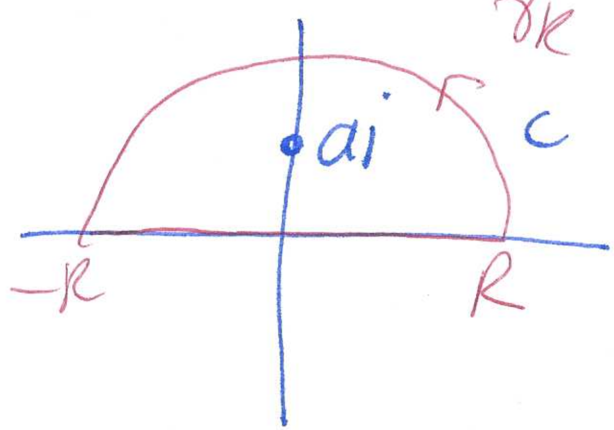
$$\left|\int_\gamma \frac{1}{(a^2+z^2)^3}\,dz\right| \le \frac{1}{(R^2-|a|^2)^3}\times \underbrace{|\gamma_R|}_{\pi R}.$$

$$\le \frac{\pi R}{(R^2-|a|^2)^3} \longrightarrow 0 \quad \text{as } R\to\infty$$

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \boxed{\int_{-\infty}^{\infty} f(x)\,dx + 0 = \frac{3\pi}{8a^5}}$$

$$\frac{1}{(a^2+x^2)^3}.$$

<u>Example</u> $\displaystyle\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx$ $\quad (a>0)$ $a\in\mathbb{R}$.

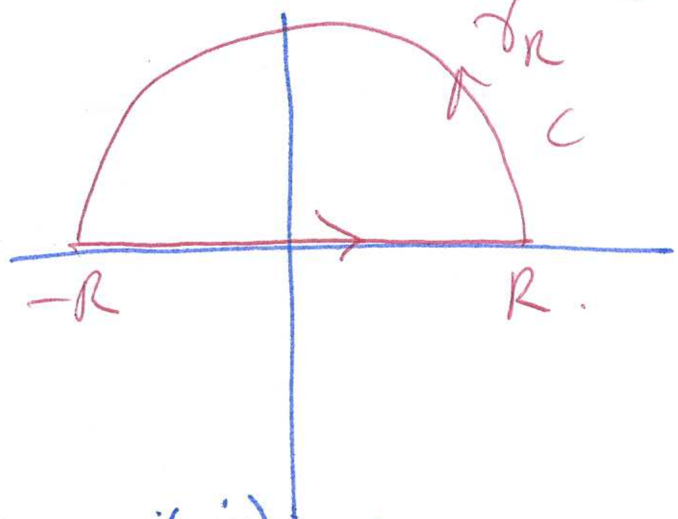


$$f(z) = \frac{e^{iz}}{z^2+a^2} \leftarrow \text{simple poles at } \pm ai$$

$$\operatorname*{Res}_{ai} f(z) = \lim_{z\to ai} \frac{(z-ai)e^{iz}}{z^2+a^2} = \lim_{z\to ai} \frac{e^{iz}}{z+ai} = \frac{e^{-a}}{2ai}$$

$$f(z) = \frac{e^{iz}}{z^2+a^2} \qquad \int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{ai} f(z) = 2\pi i \frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}$$

$\mathbb{C}$



$$\int_C f(z)\,dz$$

$$= \int_{-R}^{R} f(x)\,dx + \int_{\gamma_R} f(z)\,dz = \frac{\pi e^{-a}}{a}.$$

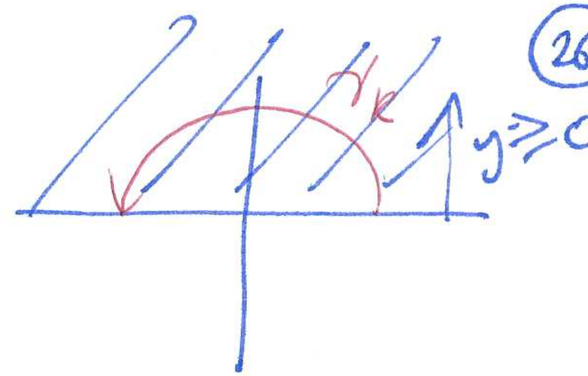
$$e^{iz} = e^{i(x+iy)} = e^{ix-y}$$

$y=0$: (on $\mathbb{R}$) $\quad e^{ix} = \cos x + i\sin x$

just look at real bit.

$$\int_{-R}^{R} \frac{\cos x}{x^2+a^2}\,dx + \underbrace{\operatorname{Re}\int_{\gamma_R} f(z)\,dz}_{} = \frac{\pi e^{-a}}{a}.$$

$$\left|\int_{\gamma_R}\frac{e^{iz}}{z^2+a^2}\,dz\right| \le \int_{\gamma_R}\left|\frac{e^{iz}}{z^2+a^2}\right|dz$$



$|z| = R.$

$$\le \frac{\pi R\, 1}{R^2-|a|^2} \to 0 \quad \text{as } R\to\infty$$

$$e^{iz} = e^{i(x+iy)} = e^{ix-y} = \underbrace{e^{-y}}_{\le 1}\underbrace{(\cos x + i\sin x)}_{\le 1}$$

$y \geq 0$

shown $\lim_{R\to\infty}\int$.

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}.$$

$|e^{iz}| \le 1.$

$$\int_{0}^{\infty}\frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2a}.$$