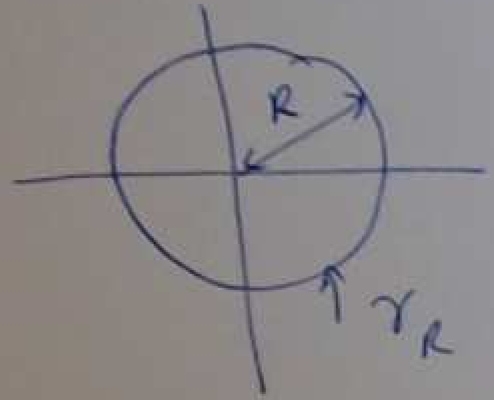
$f : \mathbb{C} \to \mathbb{C}$ cts. $\qquad M(R) = \max_{|z|=R} |f(z)|$.

$\mathbb{C}$.



$RM(R) \to 0$ as $R \to \infty$.

$$\left| \int_{\gamma_R} f(z)\,dz \right| \leq \max_{\gamma_R} |f(z)| \cdot 2\pi R.$$

$2\pi M R \to 0$ as $R \to \infty$

19. Find the rotation, magnitude and finite fixed point (if any) corresponding to each of the following entire linear transformations:
a) $w = 2z + 1 - 3i$; b) $w = iz + 4$; c) $w = z + 1 - 2i$.

20. Find the entire linear transformation with fixed point $1 + 2i$ carrying the point i into the point $-i$.

21. Find the entire linear transformation carrying the triangle with vertices at the points $0, 1, i$ into the similar triangle with vertices at the points $0, 2, 1 + i$.

22. The mapping

$$w = f(z) = \frac{az + b}{cz + d}, \tag{11}$$

where a, b, c and d are arbitrary complex numbers (except that c and d are not both zero), is called a *fractional linear transformation* (or *Möbius transformation*). Prove that

a) $f(z)$ reduces to an entire linear transformation if $c = 0$;
b) $f(z)$ reduces to a constant if $ad - bc = 0$;
c) If $c \neq 0$, $ad - bc \neq 0$, then $f(z)$ has a nonzero derivative $f'(z)$ for all z except $z = \delta = -d/c$;
d) If $c \neq 0$, $ad - bc \neq 0$, then $f(z)$ is conformal at all finite points except possibly at δ (however see Prob. 26), where the angle α through which tangents to curves are rotated has the same value

$$\alpha = \arg f'(z) = \arg \frac{ad - bc}{c^2} - 2 \arg (z - \delta)$$

along any given ray emanating from δ, and the magnification has the same value

$$\mu = |f'(z)| = \left|\frac{ad - bc}{c^2}\right| \frac{1}{|z - \delta|^2}$$

along any given circle with center δ.

23. Let μ be the same as in the preceding problem. Show that

a) $\mu = 1$ at every point of the circle with equation

$$|z - \delta| = \frac{1}{|c|}\sqrt{|ad - bc|},$$

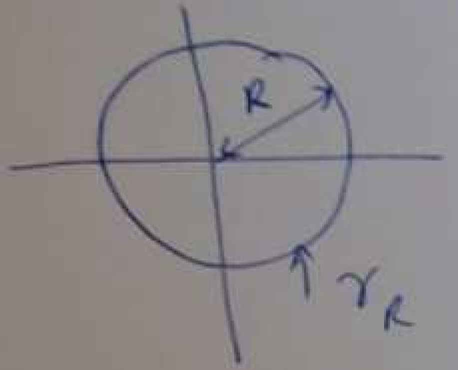
called the *isometric circle* of the transformation (11);
b) $\mu > 1$ inside γ and approaches ∞ as $z \to \delta$;
c) $\mu < 1$ outside γ and approaches 0 as $z \to \infty$.†

24. Let

$f : \mathbb{C} \to \mathbb{C}$ c/s.

$$M(R) = \max_{|z|=R} |f(z)|$$

$\mathbb{C}$.



$RM(R) \to 0$ as $R \to \infty$.

$$\left| \int_{\gamma_R} f(z)\,dz \right| \leq \max_{\gamma_R} |f(z)| \cdot 2\pi R.$$

$2\pi MR \to 0$ as $R \to \infty$

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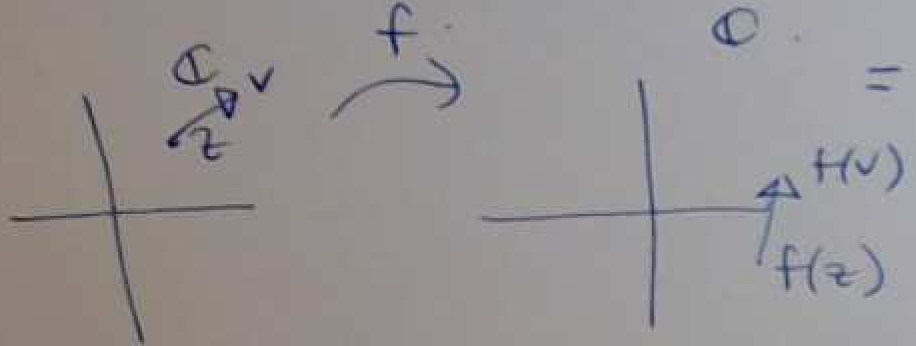
b) $\mu > 1$ inside γ and approaches ∞ as $z \to \delta$;
c) $\mu < 1$ outside γ and approaches 0 as $z \to \infty$.†

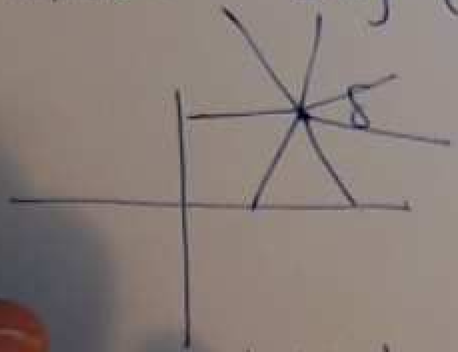
24. Let

$$f(z) = \frac{az+b}{cz+d} \qquad f'(z) = \frac{(cz+d)a - c(az+b)}{(cz+d)^2}$$

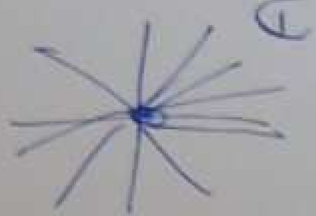


$$= \frac{ad-bc}{(cz+d)^2} = \frac{ad-bc}{c^2\left(z+d/c\right)^2}$$

rotation: $\arg\left(\frac{ad-bc}{(cz+d)^2}\right) = \arg\left(\frac{ad-bc}{c^2}\right) - 2\arg(z-\delta)$



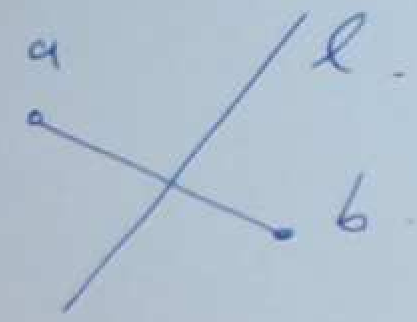
$\arg(z-\delta) = \text{const}$



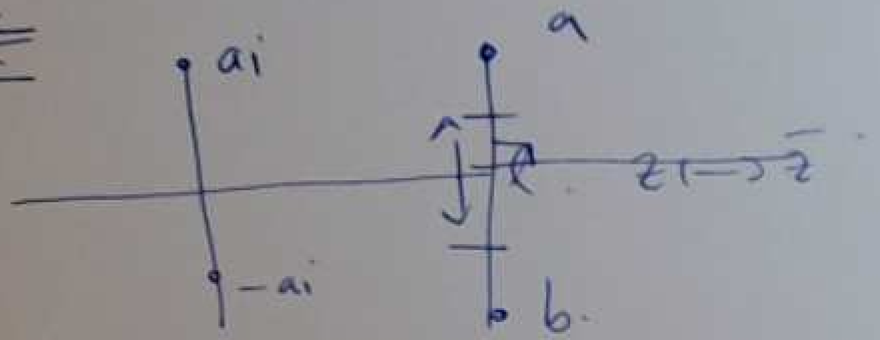
$\arg(z) = \text{const}$

$$|f'(z)| = \left|\frac{ad-bc}{c^2}\right| \frac{1}{|z-\delta|^2}$$

e) $h_a r_\ell = r_\ell h_b$ iff



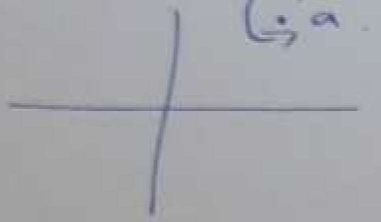
$\Leftarrow$



$h_0 : z \mapsto -z$

$\Rightarrow$ $z \mapsto \bar{z}$ $\qquad h_a$

$h_a r_\ell = z \mapsto -\bar{z} \mapsto \boxed{-\bar{z} + 2a}$



$r_\ell h_b = z \mapsto -z + 2b \mapsto \overline{-z+2b}$

$\boxed{-\bar{z} + 2\bar{b}}$

$2a + 2\bar{b} = 0$

$a + \bar{b} = 0$

$a = \bar{b}$

$x+iy$ $\quad x+iy$

$w+it$ $\quad w-it$

$z \mapsto z - a$

$z \mapsto -z$

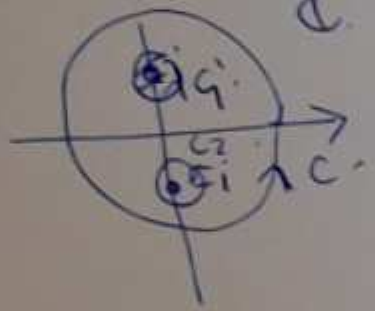
$z \mapsto z + a$

$z \mapsto z - a \mapsto -z + a \mapsto -z + 2a$

$$\int_C \frac{1}{1+z^2}\, dz.$$

(1) $\tan^{-1}(z) + c$ ✗ see later.

(2) explicit line integral.



$c(t) = \cancel{2\cos\theta}\ \ 2\cos t + i2\sin t$

$c'(t) = -2\sin t + i2\cos t$

$$\int_0^{2\pi} \frac{1}{1+c(t)^2} \cdot c'(t)\, dt \dots$$

(3) $$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$$

$$\int_{c_1} \frac{A}{z-i} + \frac{B}{z+i}\, dz = \int_{c_1} \frac{A}{z-i}\, dz + \int_{c_1} \frac{B}{z+i}\, dz \quad 0$$

$$= A 2\pi i$$

§7 Power series

<u>Defn</u> A power series is an infinite sum of the form

$$c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots$$

$c_i \in \mathbb{C}$ a called the center for the power series (usually $a=0$).

$a=0$ $\quad c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$

The <u>region of convergence</u> is the subset of $\mathbb{C}$ for which the power series converges.

<u>Fact</u> always converges at 0

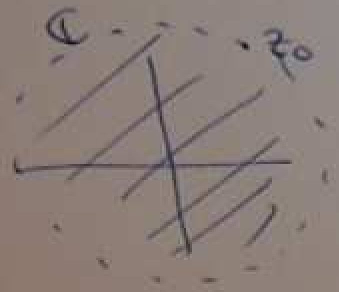
Example region of convergence may be $\{0\}$.

$1 + 1!z + 2!z + 3!z^2 + 4!z^3 + \cdots$ exercise

← check this only converges for $z = 0$.

Lemma suppose $\sum_{n=0}^{\infty} c_n z^n$ converges at $z_0 \neq 0$.

then it is absolutely convergent for $|z| < |z_0|$.



Proof suppose $c_0 + c_1 z_0 + c_2 z_0^2 + \cdots$ converges.

$\Rightarrow \lim_{n \to \infty} c_n z_0^n = 0$.

$\Rightarrow$ $c_n z_0^n$ is a bounded sequence

there is M s.t. $|c_n z_0^n| < M$ for all n.

pick $|z| < |z_0|$ $\qquad |c_n z^n| = |c_n z_0^n| \left|\frac{z}{z_0}\right|^n$.

$|c_n z^n| < M \underbrace{\left|\frac{z}{z_0}\right|}_{<1}^n$ sum $\sum |c_n z^n|$ converges as bounded above by a geometric series

$\Rightarrow$ $\sum c_n z^n$ absolutely convergent. $\square$

<u>Example</u> $1 + z + z^2 + z^3 + z^4 + \cdots$ $\qquad \frac{1}{1-z}$ $\qquad |z| < 1$

<u>Thm</u> Suppose $\sum c_n z^n$ converges for at least one $z_0 \neq 0$. Then there is $R \in \mathbb{R}$, $R > 0$ s.t. $\sum c_n z^n$ converges for all $|z| < R$, diverges $|z| > R$.

Q: don't know about $|z| = R$

<u>Proof</u> $R = \sup |z|$ over all z where $\sum a_n z^n$ converges.

if $|z| < R$, then $\sum a_n z^n$ by previous.

spose $|z| > R$ and $\sum a_n z^n$ converges # $R = \sup_{\sum a_n z^n \text{ converges}} |z|$. $\square$

<u>Example</u> ① $1 + z + z^2 + z^3 + \cdots$ $|z| < 1$ $R = 1$.

sup means supremum means least upper bound.

$E \subseteq \mathbb{R}$

$\sup(E) =$ least upper bound. $\in \mathbb{R} \cup \{+\infty\}$.

<u>Recall</u> ① uniformly convergent for $|z| \leq r < 1$

but <u>not</u> uniformly convergent for $|z| < 1$

Th^m If $\sum c_n z^n$ has radius of convergence R, (10)
then it is uniformly convergent for $|z| \leq r < R$.

Proof $|c_n z^n| = |c_n||z|^n \leq |c_n| r^n$ unif converges for all $r < R$.

point: $\sum |c_n z^n|$ uniformly convergent $\Rightarrow$ $\sum c_n z^n$ is uniformly convergent. $\square$

~~Th^m~~ **Corollary** $s(z) = \sum c_n z^n$ suppose has radius of convergence R, then $s(z)$ is analytic in $|z| < R$, and furthermore can be differentiated term by term.

$$s'(z) = c_1 + c_2 2z + c_3 3z^2 + c_4 4z^3 + \cdots$$

Proof recall $\sum c_n z^n$ uniformly convergent in $|z| \le r < R$ (11)

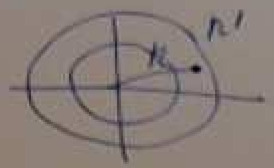
and hence its uniformly convergent in every closed bounded domain in $|z| < R$. each $c_n z^n$ is analytic in $|z| \le r$ (in fact in $\mathbb{C}$!) and so $\sum c_n z^n$ is analytic in $|z| < R$, and can be differentiated term by term.

consider $\sum c_n n z^{n-1}$ ← ~~check~~ has radius of converge R'

check $R' = R$. observe: $|c_n z^n| = \left| c_n n z^{n-1} \frac{z}{n} \right|$, $\frac{z}{n} < 1$ for $n >> 0$.

$R' \geq R$.

spse $R' > R$ ← then integrate $\sum c_n n z^{n-1}$



⇝ $s(z)$ back again.

\# $s(z)$ diverges outside $|z| = R$ □

§7.2 Radius of convergence

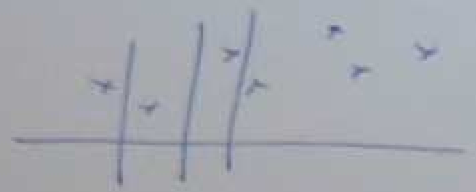
Def$^{\underline{n}}$ (a_n) is a sequence of (non-negative) real numbers.

upper limit $= \overline{\lim\limits_{n\to\infty}}\, a_n = \limsup\limits_{n\to\infty} a_n$

$=$ largest limit point of the a_n (if a_n bounded)

$= +\infty$ $\quad a_n$ not bounded.

$$= \lim_{n\to\infty} \underbrace{\sup\{a_k \mid k \geqslant n\}}_{\sigma_n}$$



$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \cdots$

decreasing seq bounded below by 0.

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(13)

$\lim_{n\to\infty} a_n$ may not exist, but $\limsup_{n\to\infty} a_n$ always exists (may be $+\infty$).

<u>Thm</u> (Cauchy - Hadamard) Let $\sum_{n=0}^{\infty} c_n z^n$ be a power series set $l = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$ then the radius of convergence R is
$$\begin{cases} +\infty & \text{if } l = 0 \\ 0 & \text{if } l = +\infty \\ 1/l & \text{otherwise} \end{cases}$$

<u>Proof</u> <u>case</u> $l = +\infty$, so $\sqrt[n]{|c_n|} \to +\infty$ want: $R = 0$

spose $\sum c_n z^n$ converges then $\lim_{n\to\infty} c_n z^n = 0$, so there is M s.t. $|c_n z^n| < M$ for all n. (assume $M > 1$)

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$|c_n z^n| < M \qquad \sqrt[n]{|c_n|\,|z|^n} \leq \sqrt[n]{M}$

$\sqrt[n]{|c_n|} < \dfrac{\sqrt[n]{M}}{|z|} \leq \dfrac{2}{|z|} \Rightarrow \sqrt[n]{|c_n|}$ bounded

$\#\ \ell = +\infty$ $\square$

<u>case 2</u> $\ell = 0$ want $R = +\infty$

$\ell = 0 \qquad \sqrt[n]{|c_n|} \to 0$ as $n \to \infty$

for any $\epsilon > 0$ there is a N s.t. $\sqrt[n]{|c_n|} < \epsilon$ for all $n \geq N$.

choose $\epsilon = \dfrac{1}{2|z|}$

$\sqrt[n]{|c_n|} < \dfrac{1}{2|z|}$

$\sqrt[n]{|c_n|}\,|z| < \dfrac{1}{2}$

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$$|c_n z^n| < \frac{1}{2^n}.$$

$\sum |c_n z^n|$ is bounded above by a geometric series and so is absolutely convergent. □

lim sup

case $\ell \neq 0, +\infty$ want $R = \frac{1}{\ell}$. for all $\epsilon > 0$, there is N s.t. $\sqrt[n]{|c_n|} < \ell + \epsilon$ for all $n > N$.

• assume $\boxed{|z| < \frac{1}{\ell}}$ set $\epsilon = \frac{1 - \ell|z|}{2|z|}$

<1.

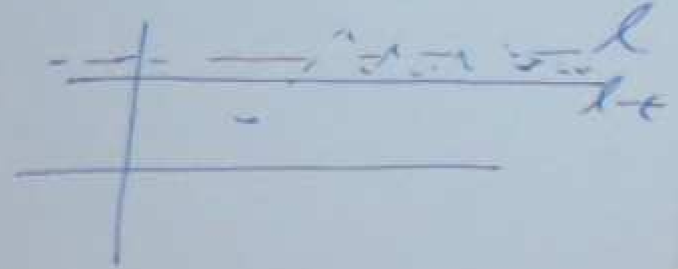
$$\sqrt[n]{|c_n|} < \ell + \epsilon \Rightarrow \sqrt[n]{|c_n|} < \ell + \frac{1 - \ell|z|}{2|z|} = \frac{1 + \ell|z|}{2|z|} < 1$$

$$\sqrt[n]{|c_n|} \leq \frac{q}{|z|} < 1. \sim |c_n z^n| < q^n$$ geometric bound

$\Rightarrow$ abs. convergent □

· pick $|z| > \frac{1}{\ell}$. for any $\epsilon > 0$ $\sqrt[n]{|c_n|} > \ell - \epsilon$

for infinitely many n.



$\epsilon = \frac{\ell|z|-1}{|z|}$.

so $\sqrt[n]{|c_n|} > \ell - \epsilon = \ell - \frac{\ell|z|-1}{|z|} = \frac{1}{|z|}$

so $|c_n z^n| > 1$ for ∞-many n, $\lim_{n\to\infty} c_n z^n \neq 0$ $\square$

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<u>Example</u> 1 $1+z+z^2+z^3+\cdots$ $c_n = 1$.

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = 1 \quad l=1 \Rightarrow R = 1.$$

2 $1 + z + z^4 + z^9 + z^{16} + z^{25} + \cdots$ $c_n = 0$ $n \neq$ square, 1 n square

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = \overline{\lim_{n\to\infty}} \{1, 0, \ldots\}$$

limit points 0, 1

$= 1$

$l=1 \Rightarrow R = 1.$

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(3) $1 + \frac{z}{1^s} + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots \quad (s \geq 0)$

$$\sqrt[n]{|c_n|} = \frac{1}{n^{s/n}} = \frac{1}{e^{s\log n/n}}$$

$\frac{\log n}{n} \to 0$ as $n \to \infty$. $e^0 = 1$. $\quad l = 1. \quad R = 1.$

(4) $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \quad l = 0 \quad R = +\infty.$

$$\sqrt[n]{|c_n|} = \sqrt[n]{1/n!}$$

note. $(n!)^2 = \underbrace{(1\,n)}_{\geq n}\underbrace{[2(n-1)]}_{\geq n}\underbrace{[3(n-2)]}_{\geq n}\cdots\underbrace{[(n-1)2]}_{\geq n}\underbrace{(n\,1)}_{\geq n} \geq n^n$

$k(n-k+1) - n \geq (k-1)(n-k) \geq 0 \qquad n! \geq \sqrt{n^n}$

$kn - k^2 + k - n$

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(19)

$$\sqrt[n]{|c_n|} = \sqrt[n]{1/n!} \le \sqrt[n]{\frac{1}{\sqrt{n}^n}} \le \frac{1}{\sqrt{n}}$$

$\to 0$ as $n \to \infty$ so $\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = 0$ $\quad l = 0$

$R = +\infty$.

(5) $1 + z + 2!z^2 + 3!z^3 + \cdots$

as above $\sqrt[n]{|c_n|} \ge \sqrt{n} \to \infty$ $\quad l = \infty$

$R = 0$.

Behaviour on $|z| = R$ can be arbitrary

$s=0$: $1 + z + z^2 + z^3 + \cdots$ $\leftarrow$ diverges on $|z| = 1$.

$s=1$: $1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$ $\leftarrow$ converges some points, diverges for some points

$s=2$: $1 + z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \cdots$ $\leftarrow$ converges on $|z| = 1$.

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