Math 231 Calculus 1 Fall 21 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $f(x) = x^2 \cos(x)$.

$$f'(x) = 2x\cos x + x^2 \cdot -\sin x$$

(b) $f(x) = \dfrac{e^x}{\ln(x)}$.

$$f'(x) = \frac{\ln(x) \cdot e^x - e^x \cdot \frac{1}{x}}{(\ln(x))^2}$$

(2) (10 points) Find the derivative of the function $f(x) = \tan^{-1}\left(2+\sqrt{x}\right)$.

$$f'(x) = \frac{1}{1+(2+\sqrt{x})^2} \cdot \frac{1}{2}x^{-1/2}$$

(3) (10 points) Find the second derivative of the function $f(x) = \sqrt{2x-3}$.

$f'(x) = \frac{1}{2}(2x-3)^{-1/2} \cdot 2 = (2x-3)^{-1/2}$

$f''(x) = -\frac{1}{2}(2x-3)^{-3/2} \cdot 2 = -(2x-3)^{-3/2}$

(4) (10 points) Use implicit differentiation to find the tangent line to the curve given by the equation $x^2 + 2y^3 = xy + 4$ at the point $(2, -1)$.

$2x + 6y^2 y' = y + xy'$

$(x,y) = (2,-1)$: $4 + 6y' = -1 + 2y'$

$4y' = -5$ $y' = -5/4$.

$y + 1 = -\frac{5}{4}(x-2)$

(5) Find the following limit: $\lim_{x\to 0} \dfrac{e^{3x^2}-1}{\sin^2(2x)}$

$$\overset{l'H}{=} \lim_{x\to 0} \frac{e^{3x^2} \cdot 6x}{2\sin(2x)\cos(2x)\cdot 2} = \lim_{x\to 0} \frac{6x\, e^{3x^2}}{2\sin(4x)}$$

$$\overset{l'H}{=} \lim_{x\to 0} \frac{6e^{3x^2} + 6x e^{3x^2}\cdot 6x}{2\cos(4x)\cdot 4} = \frac{6}{8} = \frac{3}{4}$$

(6) (10 points) An oil tanker is leaking oil and forming a circular oil slick. If the area of the oil slick is growing at a rate of $5\text{m}^2/\text{minute}$, how fast is the radius growing when the radius is 4m? (The area of a circle is $A = \pi r^2$.)

$A(t) = \pi r(t)^2$

$\frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt}$ $\quad$ $\frac{dA}{dt} = 5, \; r = 4$ $\quad$ $5 = \pi \cdot 8 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{8\pi}$

(7) (10 points) Use linear approximation to estimate $\sqrt{98}$. What is the percentage error in your approximation?

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f(98) \simeq f(100) + f'(100) \cdot (98-100)$

$= 10 + \frac{1}{20} \cdot -2$

$\approx 10 - \frac{1}{10} = 9.9$

percentage error : $\dfrac{|9.9 - \sqrt{98}|}{\sqrt{98}} \times 100 \approx 0.0051\%$

(8) Find the critical points for the function $f(x) = x^3 - 12x$ and use the first derivative test to classify them.

$f'(x) = 3x^2 - 12$

$= 3(x-2)(x+2)$

Solve $f'(x) = 0$:

$3x^2 - 12 = 0$

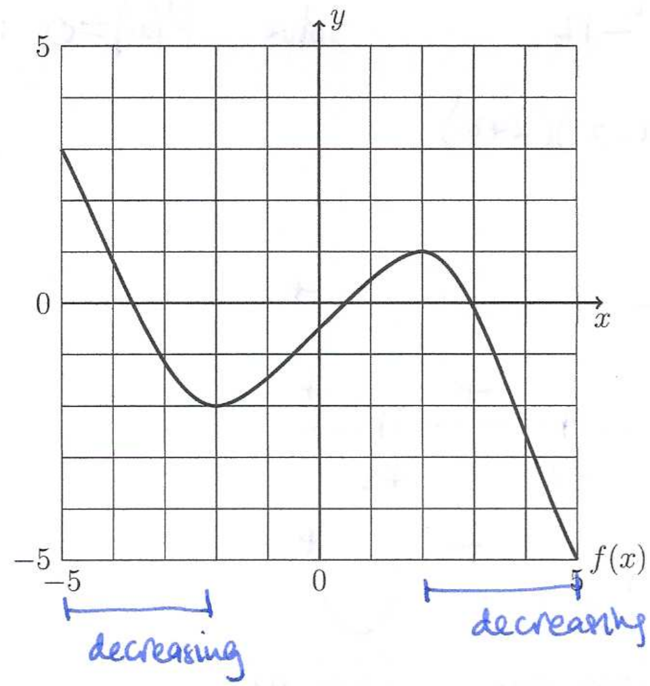
$x^2 = 4$

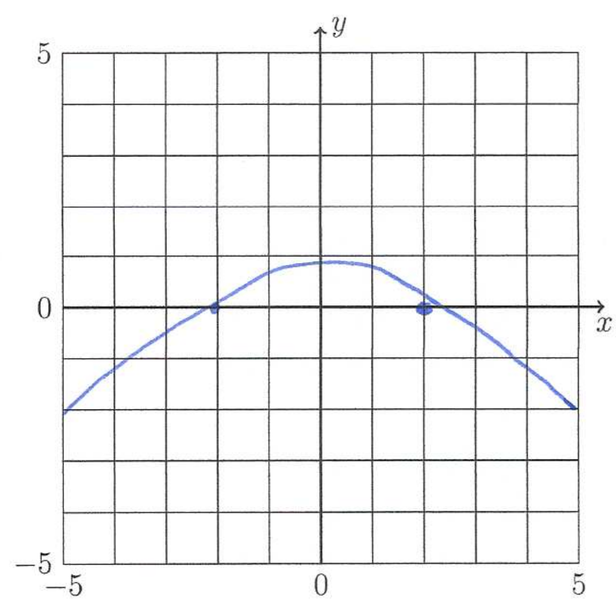
$x = \pm 2$

		-2		$+2$	
$(x-2)$	$-$		$-$		$+$
$(x+2)$	$-$		$+$		$+$
$f'(x)$	$+$		$-$		$+$

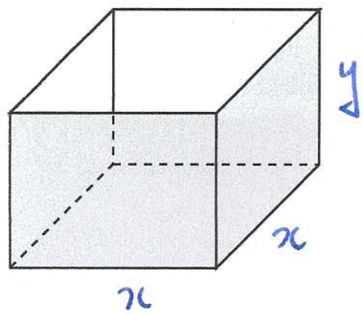
local max   local min

(9) (10 points) The graph of the function $f(x)$ is shown below. On the top set of axes mark where $f(x)$ is decreasing. On the lower set of axes sketch $f'(x)$.





(10) A container consists of a square base and four vertical sides, but without a top side. If the total area of the container is 1m^2, what is the largest possible volume of the container?



$$V = x^2 y$$

$$A = x^2 + 4xy = 1$$

$$y = \frac{1-x^2}{4x}$$

$$V = x^2 \cdot \frac{(1-x^2)}{4x} = \frac{1}{4}(x - x^3)$$

$$\frac{dV}{dx} = \frac{1}{4}(1-3x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{3}}.$$

$$y = \frac{1-\frac{1}{3}}{4\,{}^1/\sqrt{3}} = \frac{{}^2/_3}{{}^4/\sqrt{3}} = \frac{2}{3} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{6}$$

$$V = \frac{1}{3} \cdot \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{18}$$