SMT 1 Solutions

Q1 a) $\frac{3}{2}$ b) 2 c) DNE d) -1 e) -1 f) 3

Q2 a) $\lim\limits_{x\to -2} \dfrac{x^2-x-6}{x+2} = \lim\limits_{x\to -2} \dfrac{(x+2)(x-3)}{(x+2)} = \lim\limits_{x\to -2} x-3 = -5$

b) $\lim\limits_{x\to \infty} \dfrac{\sqrt{3+2x^2}}{2-x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{2+3/x^2}}{-1+2/x} = -\sqrt{2}$

c) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{5x} \overset{\theta = 4x}{=} \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{5\theta/4} = \dfrac{4}{5}\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = \dfrac{4}{5}.$

d) $\lim\limits_{x\to 0+} \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{x+1}} = \lim\limits_{x\to 0+} \dfrac{(\sqrt{x+1}-\sqrt{x})}{\sqrt{x(x+1)}}\dfrac{(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} = \dfrac{\cancel{x}+1-\cancel{x}}{\sqrt{x(x+1)}(\sqrt{x+1}+\sqrt{x})}. = 0.$

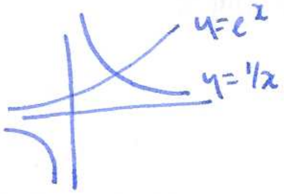
Q3 $\lim\limits_{x\to 3-} f(x) = \lim\limits_{x\to 3-} 2x - \dfrac{1}{x-2} = 6-1 = 5$

$\lim\limits_{x\to 3+} f(x) = \lim\limits_{x\to 3+} \dfrac{\cos(\pi x)}{x} = \dfrac{-1}{3}$

$\Big\}$ no value of c makes this c/s $5 \neq -1/3$.

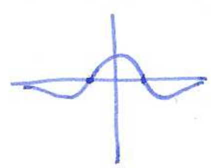
Q4 $V = \frac{4}{3}\pi r^3$ $\quad \dfrac{1}{2}\displaystyle\int_2^4 \frac{4}{3}\pi r^3\,dr = \frac{2}{3}\left[\pi\frac{r^4}{4}\right]_2^4 = \frac{\pi}{6}(4^4-2^4).$

Q5



consider $f(x) = e^x - \frac{1}{x}$. $\quad f(1/2) = e^{1/2} - 2 < 0$, $f(2) = e^2 - \frac{1}{2} > 0$ $\Big\}$ IVT $\Rightarrow \exists x$ s.t. $f(x) = 0$.

Q6 a) $(-5,-1)\cup(1,5)$ b) $(-1,1)$ c)



d) 0 e) 2

Q7 a) $f(x) = \dfrac{2}{x-1}$ $\quad f'(3) = \lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0}\dfrac{\frac{2}{3+h-1} - \frac{2}{3-1}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{2}{2+h}-1}{h} = \lim\limits_{h\to 0} \dfrac{1}{h}\dfrac{2-(2+h)}{2+h} = \lim\limits_{h\to 0}\dfrac{-h}{h(2+h)} = \lim\limits_{h\to 0} -\dfrac{1}{2+h} = -\dfrac{1}{2}$

b) $f(x) = \dfrac{1}{\sqrt{x+1}}$ $\quad f'(3) = \lim\limits_{h\to 0} \dfrac{\frac{1}{\sqrt{3+h+1}} - \frac{1}{\sqrt{4}}}{h} = \lim\limits_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{4+h}} - \dfrac{1}{2}\right) = \lim\limits_{h\to 0}\dfrac{1}{h}\dfrac{2-\sqrt{4+h}}{2\sqrt{4+h}}.$

$= \lim\limits_{h\to 0}\dfrac{1}{h}\dfrac{2-\sqrt{4+h}}{2\sqrt{4+h}}\cdot\dfrac{2+\sqrt{4+h}}{2+\sqrt{4+h}} = \lim\limits_{h\to 0}\dfrac{4-(4+h)}{h.2\sqrt{4+h}(2+\sqrt{4+h})} = \lim\limits_{h\to 0}\dfrac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})} = \dfrac{-1}{16}$

~~Q8~~ Q9

a) $f(x) = 2x^2 - \cancel{8}x^3 - 2\sqrt{x} + 4x^{-2/3}.$

$f'(x) = 4x - 3x^2 - x^{-3/2} - \frac{8}{3}x^{-5/3}$

$f''(x) = 4 - 6x + \frac{3}{2}x^{-5/2} + \frac{40}{9}x^{-8/3}$

b) $f(x) = 2x^3e^{\cancel{-4}x}$ $\quad f'(x) = 6xe^x + 2x^3e^x$

$f''(x) = 6e^x + 6xe^x + 6x^2e^x + 2x^3e^x$

c) $f(x) = \frac{2x+3}{2x-3}$ $\quad f'(x) = \frac{(2x-3)(2x+3)' - (2x-3)'(2x+3)}{(2x-3)^2} = \frac{4x-6-4x-6}{(2x-3)^2}$

$f'(x) = \frac{-12}{(2x-3)^2}$ $\quad f''(x) = \frac{(2x-3)^2\cdot(-12)' - ((2x-3)^2)'\cdot -12}{(2x-3)^4}$

$f''(x) = \frac{12(4x^2-12x+9)'}{(2x-3)^4} = \frac{12(8x-12)}{(2x-3)^4} = \frac{48(2x-3)}{(2x-3)^4} = \frac{48}{(2x-3)^3}$

d) $f(x) = \frac{\sqrt{2x}-1}{3-\cos(x)}$ $\quad f'(x) = \frac{(3-\cos(x))(\sqrt{2}\sqrt{x}-1)' - (3-\cos(x))'(\sqrt{2x}-1)}{(3-\cos(x))^2}$

$f'(x) = \frac{(3-\cos(x))(\sqrt{2}\cdot\frac{1}{2}x^{-1/2}) - \sin(x)(\sqrt{2x}-1)}{(3-\cos(x))^2}$

$f''(x)$ = omit this one.

e) $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ $\quad f'(x) = \frac{\cos(x)\cdot(\sin x)' - (\cos x)'\cdot\sin x}{\cos(x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$f'(x) = \sec^2(x)$ $\quad f''(x) = \cancel{\frac{1}{\cos^2 x} = \frac{\cos^2 x(1)' - (\cos^2 x}{\cos^4 x}}$ (*) ~~omit this one~~

f) $\cancel{f(x) = \mathrm{cosec}(x) = \frac{1}{\sin x}\quad f'(x) =}$ $\quad f(x) = \sin^2(x) = \sin(x)\sin x$

$f'(x) = (\sin x)'\sin x + \sin x(\sin x)' = 2\sin x\cos x$

$f''(x) = 2(\sin x)'\cos x + 2\sin x(\cos x)' = 2\cos^2 x - 2\sin^2 x$

(*) e) $f''(x) = \frac{\cos^2 x(1)' - (\cos^2 x)'\cdot 1}{\cos^4 x} = \frac{-(\cos x\cdot\cos x)'}{\cos^4 x} = -\frac{(\cos x)'\cos x + \cos x(\cos x)'}{\cos^4 x}$

$= \frac{2\sin x\cos x}{\cos^4 x} = \frac{2\sin x}{\cos^3 x} = 2\tan x\sec^2 x.$

Q10 a) $h(x) = f(x)\,g(x)$ $\qquad h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

$$h'(2) = f'(2)\,g(2) + f(2)\,g'(2) = (-1)\left(-\tfrac{1}{2}\right) + (\cancel{4/2}\ 2)\cdot\tfrac{1}{2} = 3/2$$

~~b) h(x) = f(g~~

b) $h(x) = f(x)/g(x)$ $\qquad h'(x) = \dfrac{g(x)\,h'(x) - g'(x)\,h(x)}{g(x)^2}$

$$h'(-4) = \frac{g(-4)\,f'(-4) - g'(-4)\,f(-4)}{g(-4)^2} = \frac{\frac{1}{2}\cdot(-2) - \left(-\frac{1}{2}\right)(-2)}{(1/2)^2}$$

$$= \frac{-1-1}{1/4} = -8$$