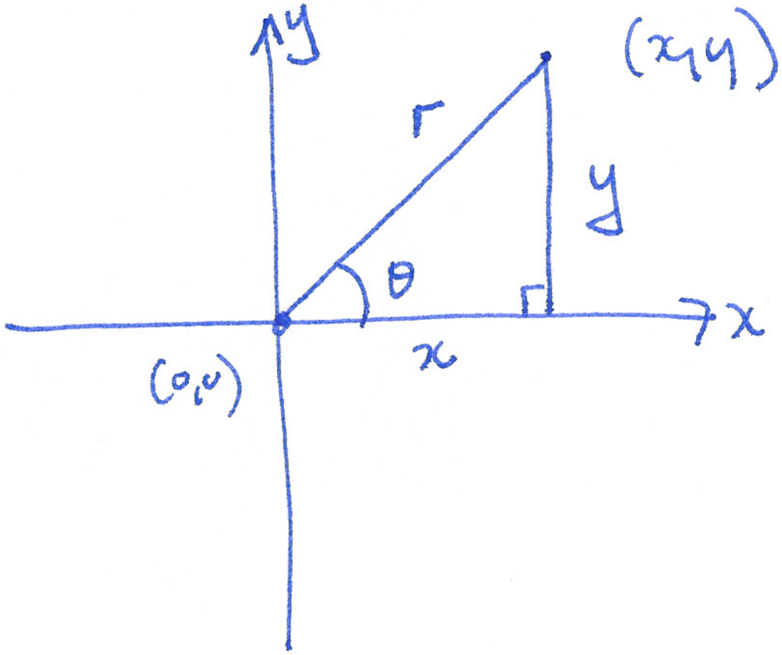
recall polar coordinates.

(r, θ).

(x, y).



$x = r\cos\theta$

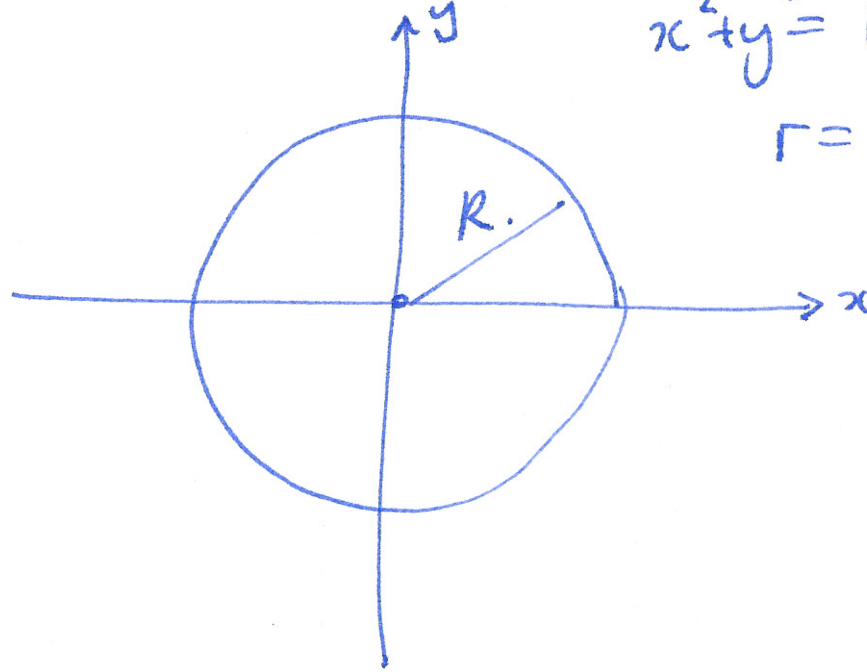
$y = r\sin\theta$

$\tan\theta = y/x$

$r^2 = x^2 + y^2$

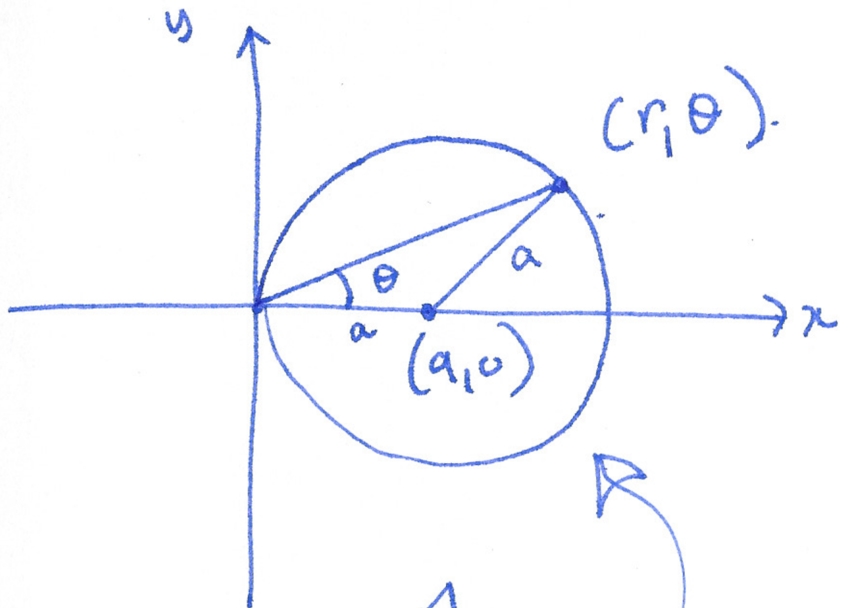
$x^2 + y^2 = R^2$.

$r = R$.



$x = r\cos\theta$

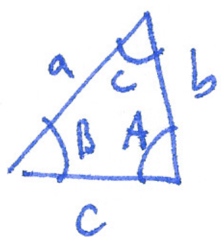
$y = r\sin\theta$



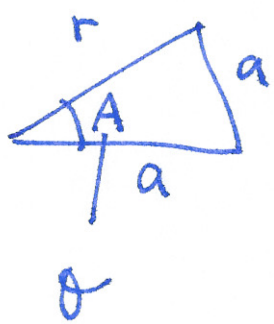
cartesian

$$(x-a)^2 + y^2 = a^2.$$

cosine rule



$$a^2 = b^2 + c^2 - 2bc\cos A.$$



$$\cancel{a^2} = r^2 + \cancel{a^2} - 2ar\cos\theta$$

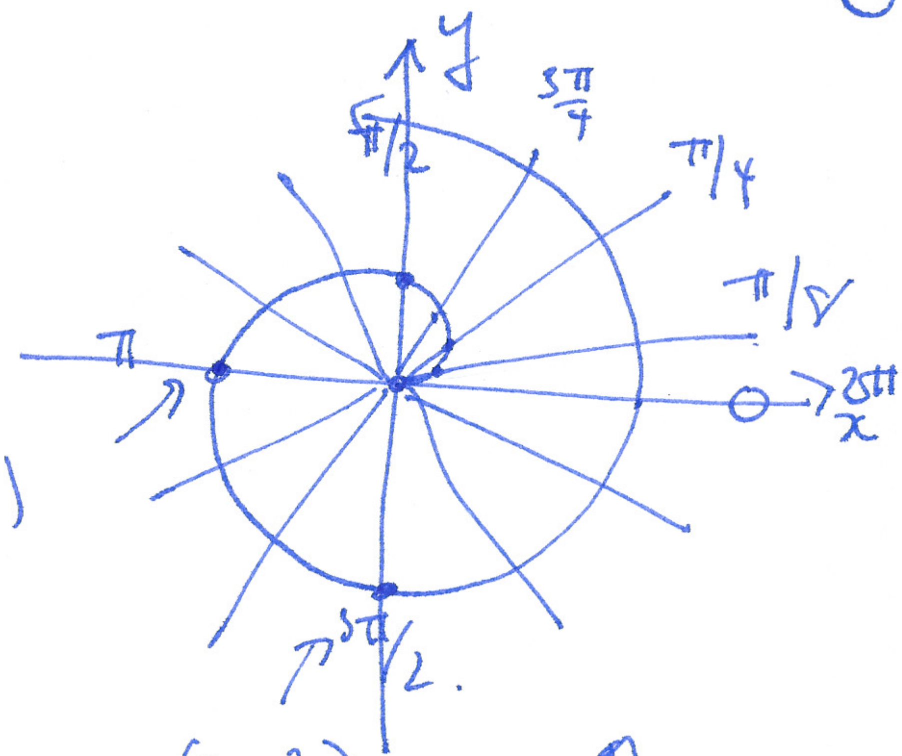
$$2ar\cos\theta = r^2.$$

$$\boxed{r = 2a\cos\theta}$$

Example $\quad r(\theta) = \theta$

$(r, \theta) \qquad (r(\theta), \theta).$

(π, π)



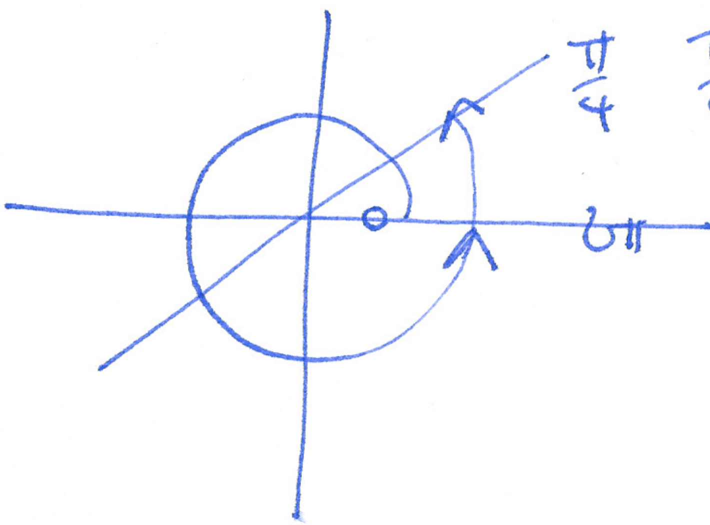
(π, π)

$(r, \theta).$

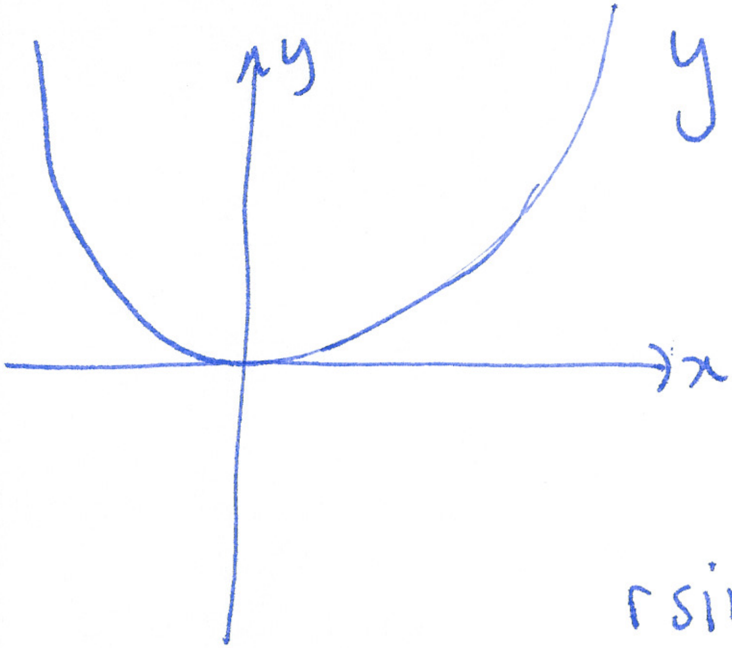
$\boxed{0 \le \theta \le 2\pi}$ X.

$\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right).$

$\theta \in [0, \infty).$



$\frac{\pi}{4} \qquad \frac{\pi}{4} + 2\pi.$



$y = x^2$.

$x = r\cos\theta$

$y = r\sin\theta$.

$$y = x^2.$$

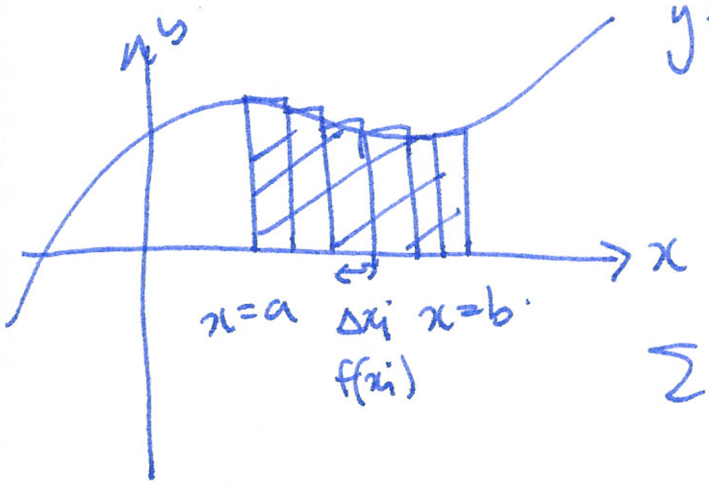
$$r\sin\theta = (r\cos\theta)^2.$$

$$r\sin\theta = r^2\cos^2\theta.$$

$$\sin\theta = r\cos^2\theta.$$

$$r = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta.$$

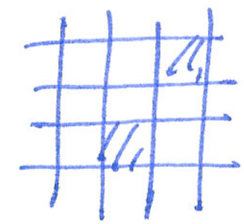
§ 11.4 Area and arc length in polars.

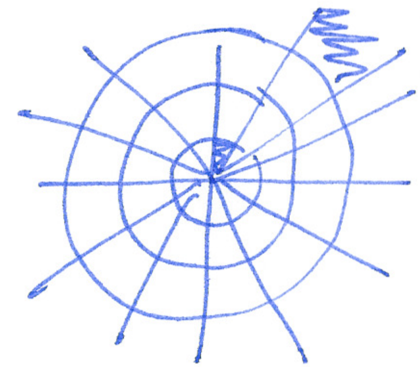


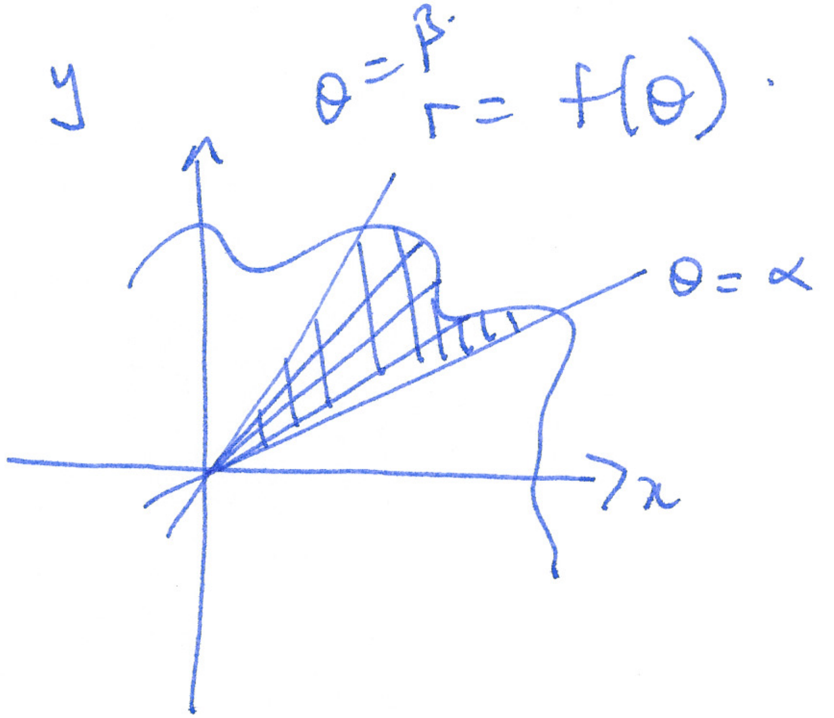
$y = f(x)$.

$$\int_a^b f(x)\,dx.$$

$\sum f(x_i)\Delta x_i$



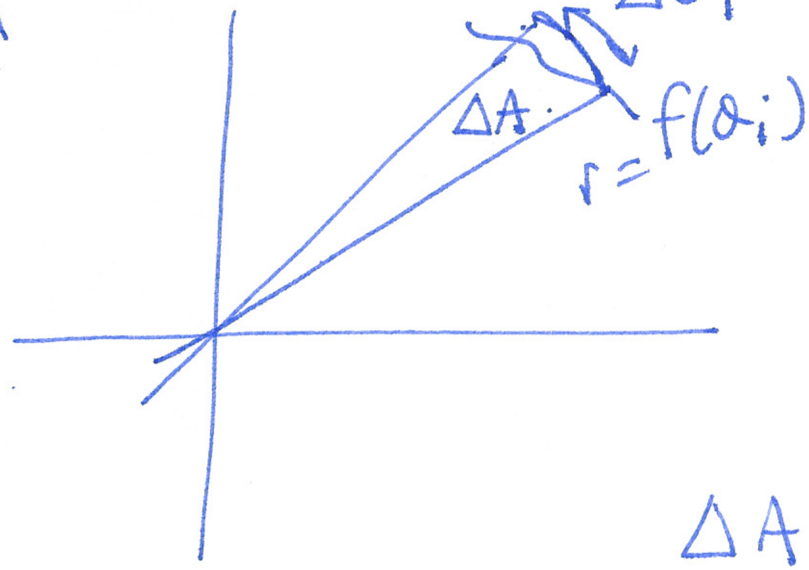




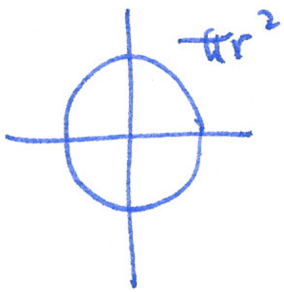
Fact.

$$\boxed{\text{area} = \int_\alpha^\beta \frac{1}{2}\left(f(\theta)\right)^2 d\theta}.$$

intuition:



$$\Delta A \approx \frac{\pi r^2}{2\pi/\Delta\theta_i} = \frac{1}{2} r^2 \Delta\theta_i$$

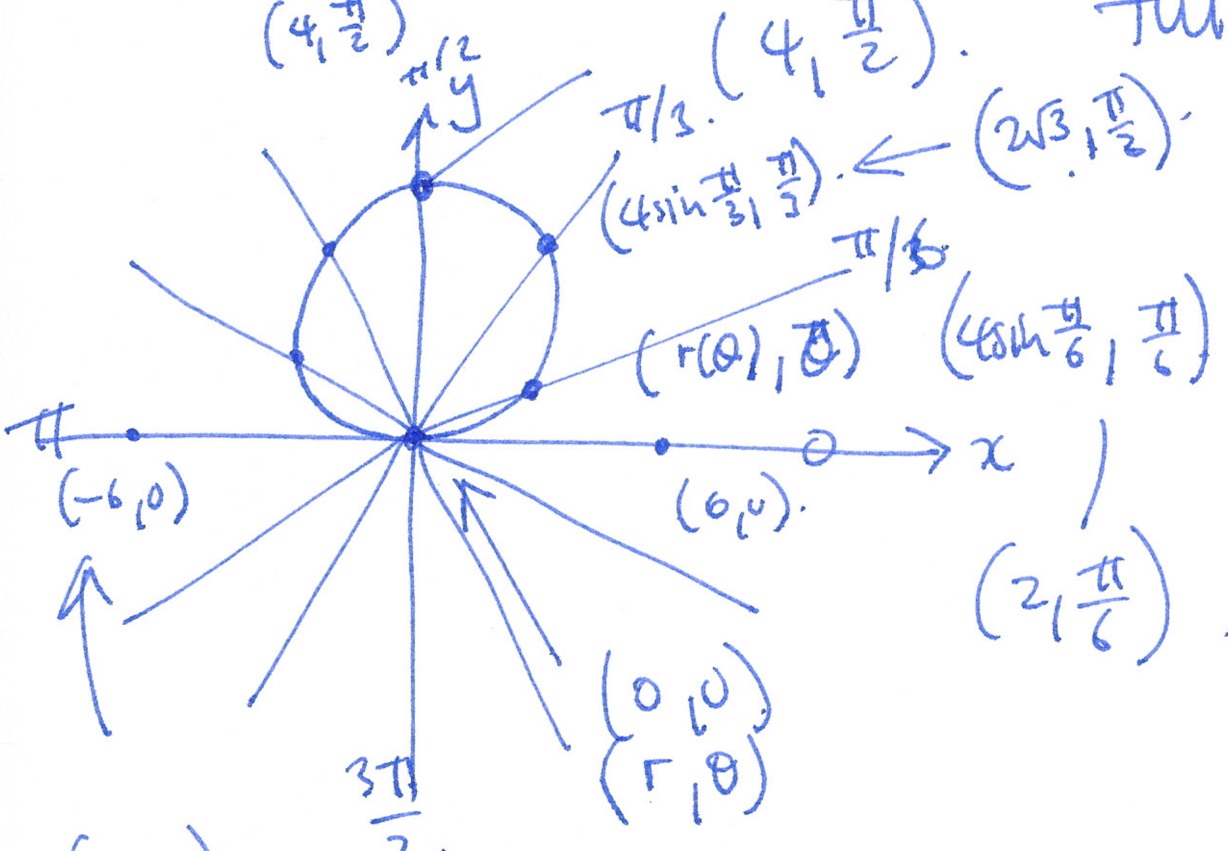


$$\Delta A_i \approx \frac{1}{2} (f(\theta_i))^2 \Delta\theta_i$$

$$\text{area} \approx \sum \Delta A_i \approx \sum \frac{1}{2} (f(\theta_i)) \Delta\theta_i$$

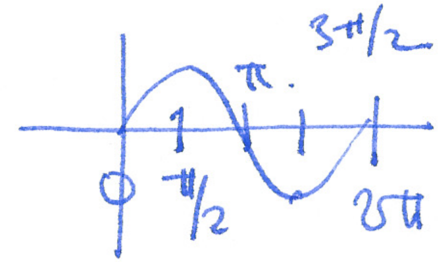
$$= \int_\alpha^\beta \frac{1}{2} (f(\theta))^2 d\theta .$$

Example $r = 4\sin\theta$ find area enclosed by this.



$(6, \pi)$

$(-6, 0)$



convention

$$(-r, \theta) = (r, \theta + \pi).$$

find area.

$$\int_0^{\pi} \frac{1}{2}\Big(\underset{4\sin\theta}{f(\theta)}\Big)^2 \, d\theta .$$

$$\int_0^{\pi} \frac{1}{2}(4\sin\theta)^2 \, d\theta = 8\int_0^{\pi} \sin^2\theta \, d\theta .$$

$$= 4\int_0^{\pi} (1-\cos 2\theta)\, d\theta .$$

$$4\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} .$$

$$4\pi .$$

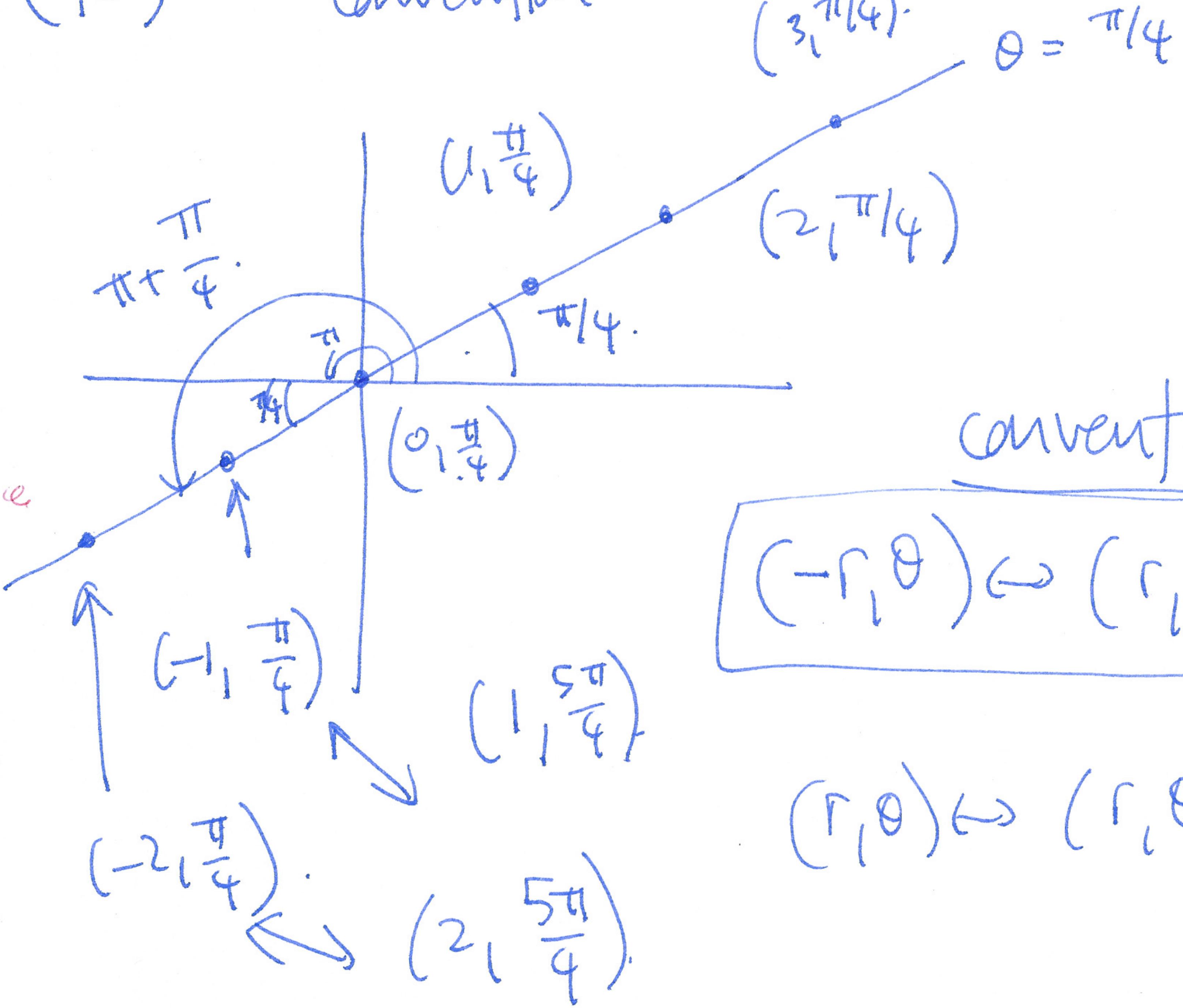
$$\cos 2\theta = \underset{1-\sin^2\theta}{\underset{\parallel}{\cos^2\theta}} - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta .$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$= \frac{1}{2}(1-\cos 2\theta)$$

(r, θ) convention.

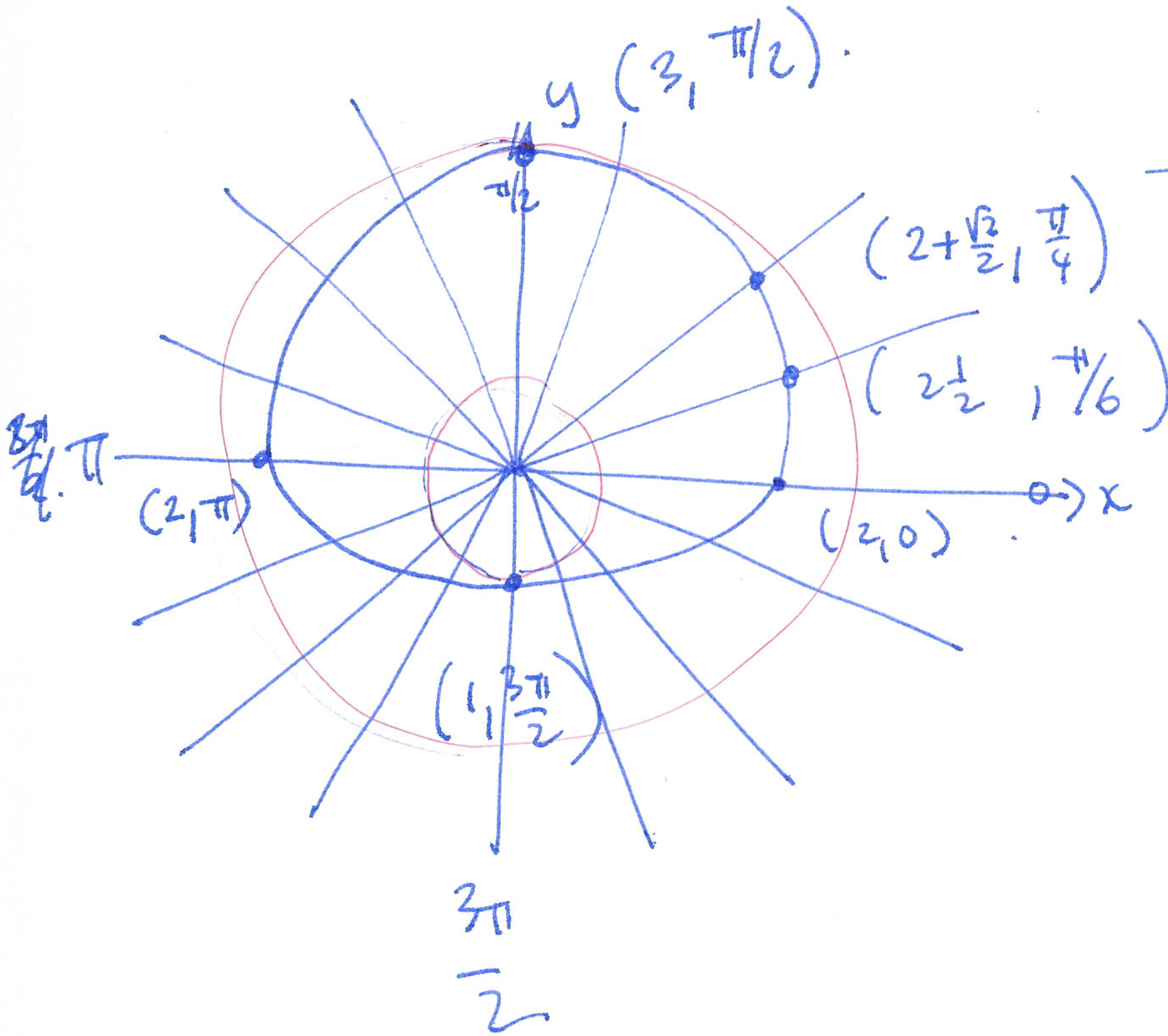


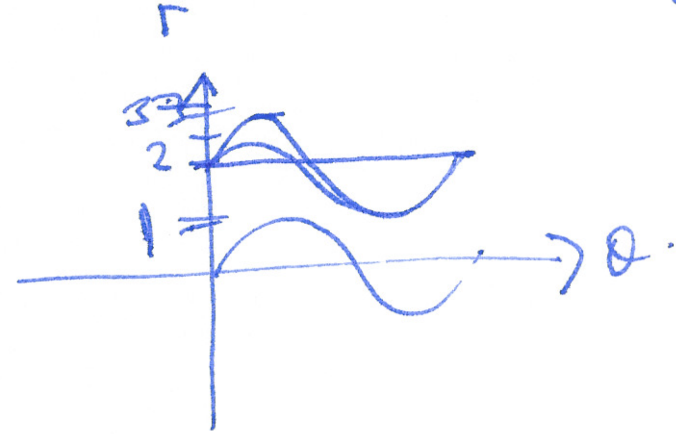
convention.

$$\boxed{(-r, \theta) \leftrightarrow (r, \theta + \pi)}$$

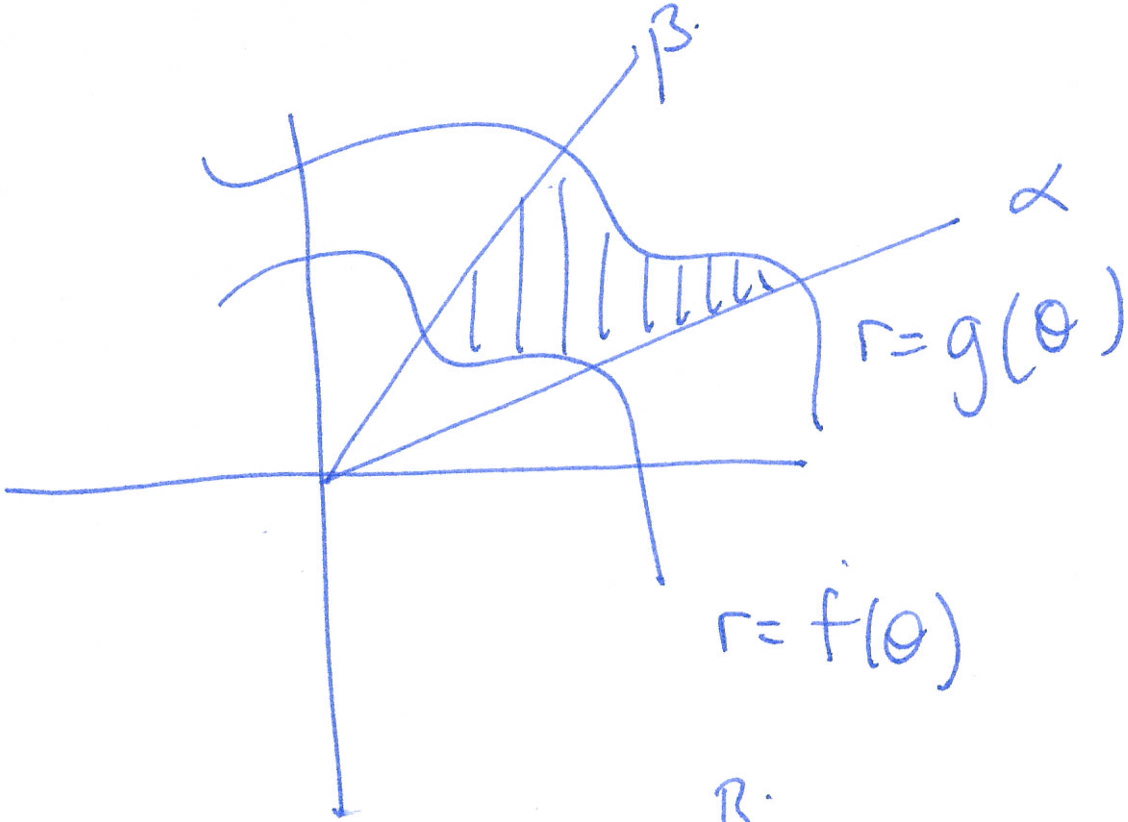
$$(r, \theta) \leftrightarrow (r, \theta + 2\pi)$$

Example. $r = 2 + \sin\theta$.



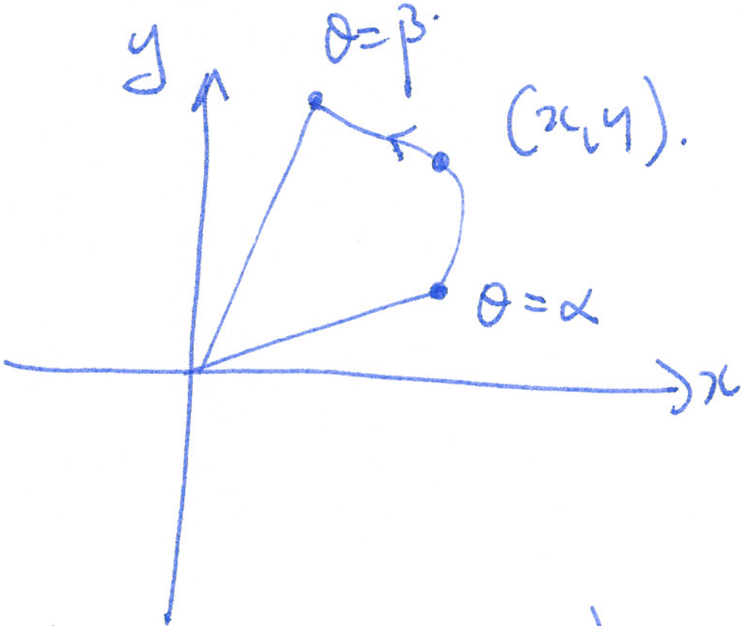


area between two curves.



$$\text{area} = \frac{1}{2}\int_{\alpha}^{\beta} (g(\theta))^2 - (f(\theta))^2 \, d\theta.$$

arc length in polars.



$r=f(\theta)$.

↑

this is a parameterized curve.

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$.

$c(\theta) = (x(\theta), y(\theta))$

arc length

$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

$x = f(\theta)\cos\theta$

$y = f(\theta)\sin\theta$

$$\boxed{\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.}$$

$$\Downarrow$$

$$\int_{\alpha}^{\beta} \sqrt{(f')^2 + (f)^2}\, d\theta.$$

$$\frac{dx}{d\theta} = f'\cos\theta \;\cancel{+}\!- f\sin\theta$$

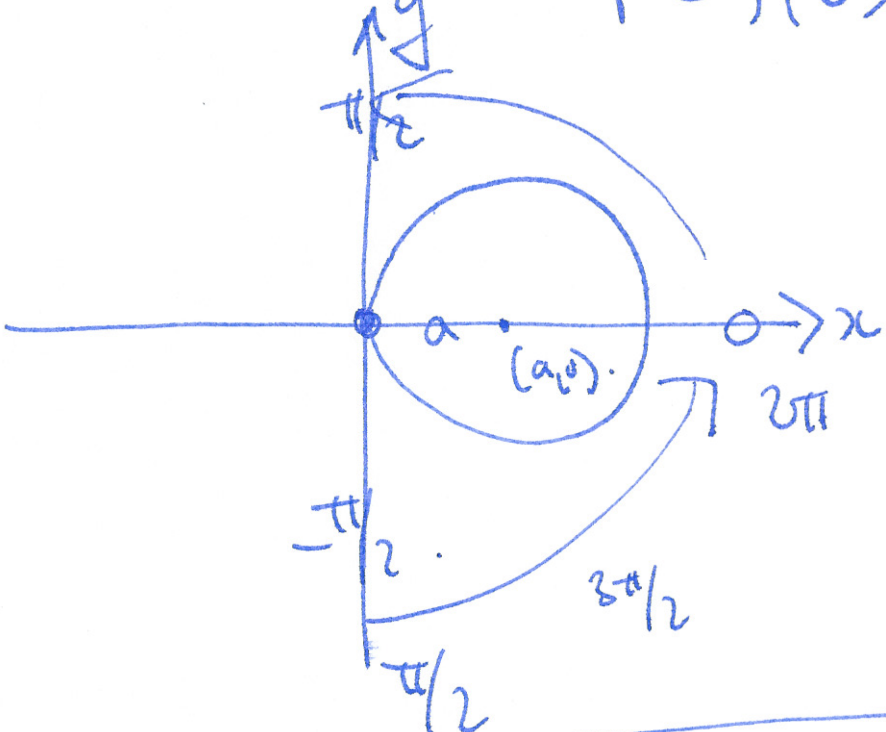
$$\frac{dy}{d\theta} = f'\sin\theta + f\cos\theta.$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(f'\cos\theta - f\sin\theta\right)^2 + \left(f'\sin\theta + f\cos\theta\right)^2$$

$$= \left.\begin{matrix} f'^2\cos^2\theta - \cancel{2f'f\cos\theta\sin\theta} + f^2\sin^2\theta \\ f'^2\sin^2\theta + \cancel{2f'f\sin\theta\cos\theta} + f^2\cos^2\theta \end{matrix}\right\}. \; f'^2 + f^2$$

Example. $r = 2a\cos\theta$. circle of radius a at $(a,0)$.

$r = f(\theta)$



$f(\theta) = 2a\cos\theta$.

$f'(\theta) = -2a\sin\theta$.

$\int_0^{\pi/2} \text{—} + \int_{\frac{3\pi}{2}-2\pi}^{2\pi-2\pi} \text{—}$ $\qquad y = f(x)$.

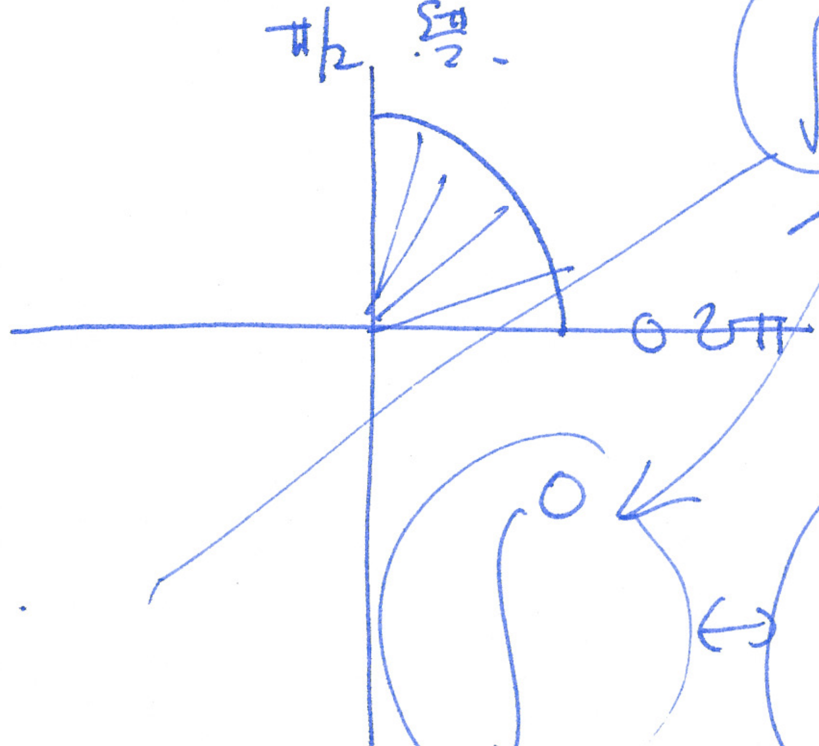
$\int_{-\pi/2}^{0}$.

arc length $= \displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{(f')^2 + (f)^2}\, d\theta$

$= \displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{4a^2\sin^2\theta + 4a^2\cos^2\theta}\, d\theta$.

$$\int_{-\pi/2}^{\pi/2}\sqrt{4a^2}\,d\theta = \int_{-\pi/2}^{\pi/2} 2a\,d\theta = \Big[2a\theta\Big]_{-\pi/2}^{\pi/2}.$$

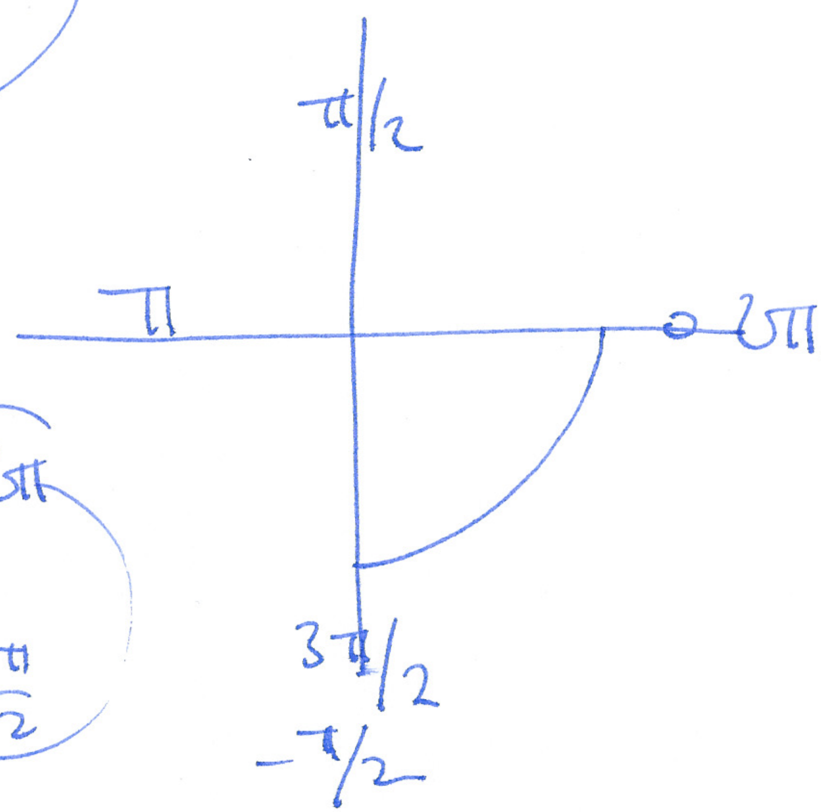
$$= 2a\frac{\pi}{2} - -2a\frac{\pi}{2} = 2\pi a.$$

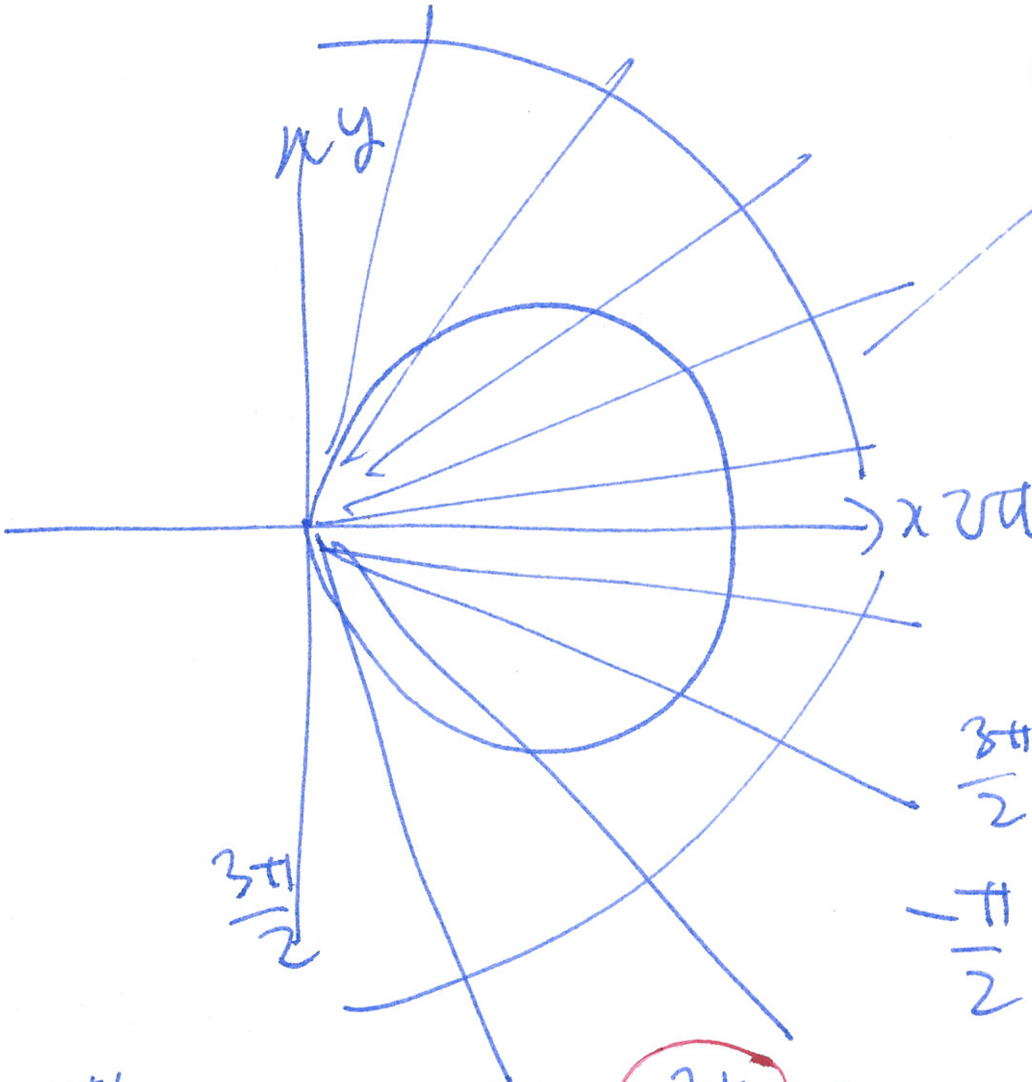


$$\int^{\pi/2}$$

$$\int_{-\pi/2}^{0} \longleftrightarrow \int_{3\pi/2}^{2\pi}$$

$$\int_{2\pi}^{5\pi/2}.$$





$0 \le \theta \le \pi/2$.

$r(\theta) = 2a\cos\theta$

$\frac{3\pi}{2} \le \theta \le 2\pi$

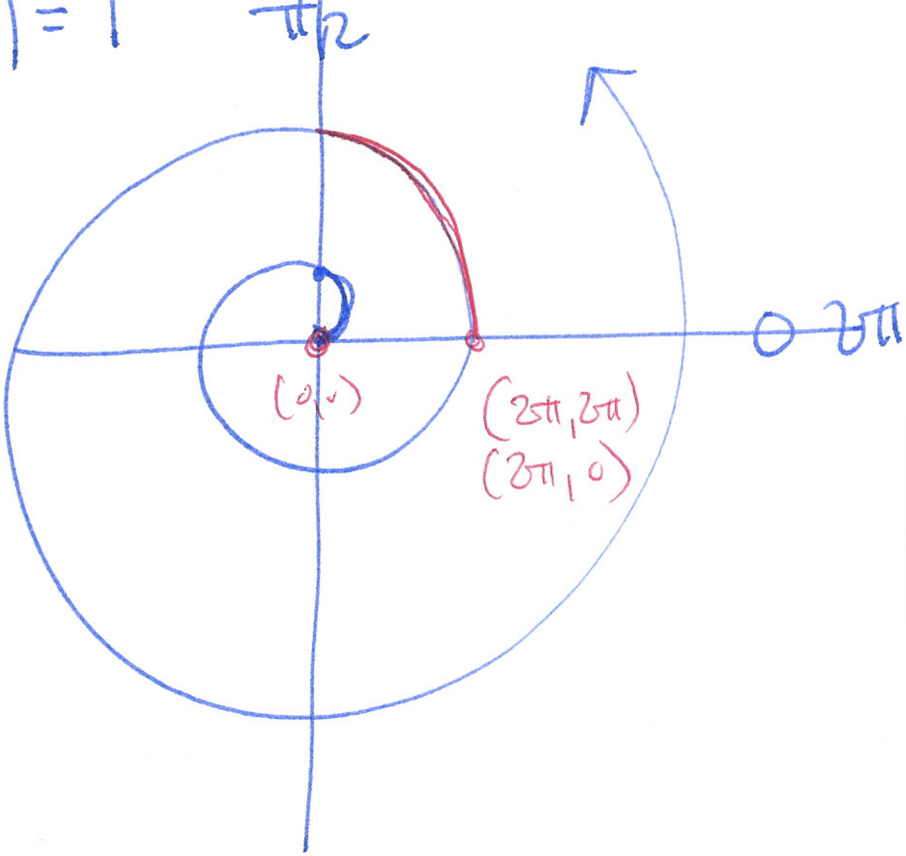
$-\frac{\pi}{2} \le \theta \le 0$.

$$\int_0^{\pi/2} \underline{\qquad\qquad} d\theta + \int_{\frac{3\pi}{2}\quad -\pi/2}^{2\pi\quad 0} \underline{\qquad\qquad} d\theta .$$

$$\int_{-\pi/2}^{\pi/2} \underline{\qquad\qquad} d\theta$$

$f(\theta) = \theta$.

$f'(\theta) = 1$.

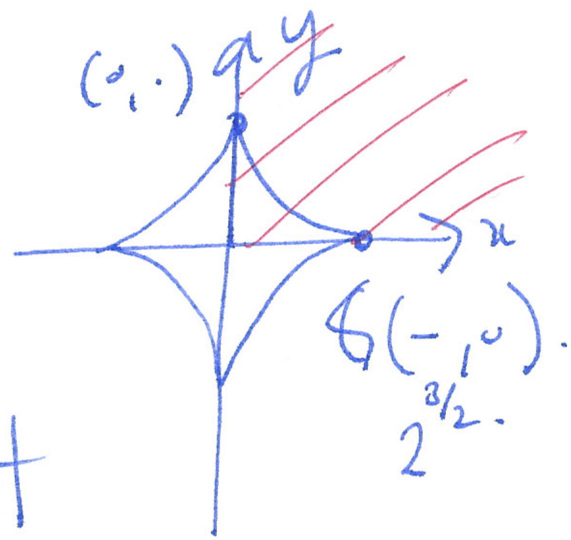


arc length

$$\int_0^{\pi/2} \sqrt{1+\theta^2}\, d\theta.$$

$$\boxed{\int_{2\pi}^{2\pi+\frac{\pi}{2}} \sqrt{1+\theta^2}\, d\theta}$$

$x^{2/3} + y^{2/3} = 2$.



arc length in first quadrant

$$\int \sqrt{(y')^2 + \cancel{(y)^2}\, 1}\; dx.$$

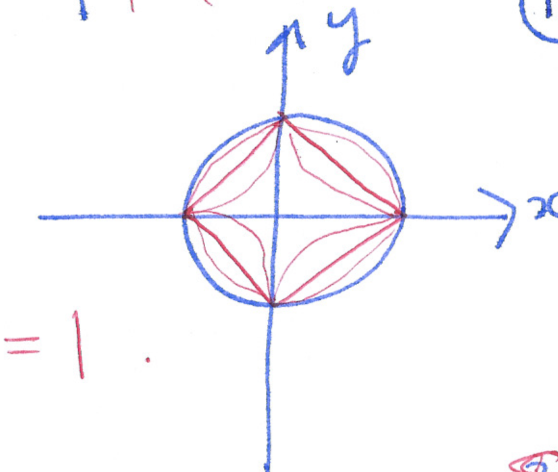
$$y^{2/3} = 2 - x^{2/3}.$$

$$y = \left(2 - x^{2/3}\right)^{3/2}.$$

$$y' = \frac{3}{2}\left(2 - x^{2/3}\right)^{1/2} \cdot -\frac{2}{3}\, x^{-1/3}.$$

$x^2 + y^2 = 1$ p=2

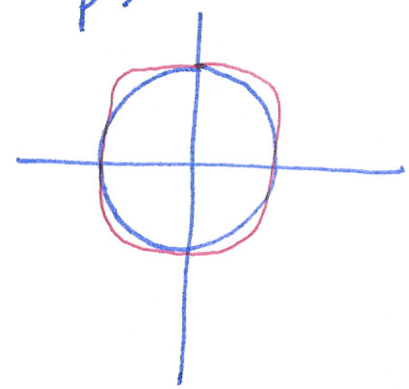
$x^p + y^p = 1$



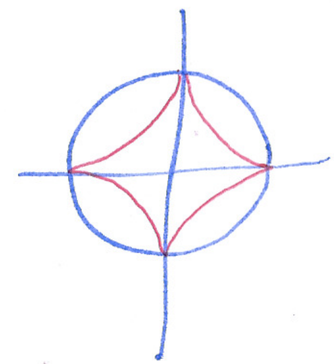
p=1.

$|x| + |y| = 1$.

p > 2



p < ~~2~~ 1

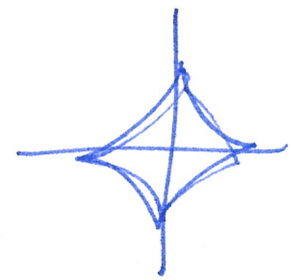


$$y = (2 - x^{2/3})^{3/2} \qquad y' = -(2 - x^{2/3})^{1/2} x^{-1/3}.$$

$$\int_0^{2^{3/2}} \sqrt{y'^2 + \cancel{y^2}\,1}\; dx$$

$$= \int_0^{2^{3/2}} \sqrt{(2 - x^{2/3})\, x^{-2/3} + \cancel{(2 - x^{2/3})^{3}}\,1}\; dx.$$

$$= \int_0^{2^{3/2}} \sqrt{2x^{-2/3} \cancel{-1} + \cancel{1}}\; dx$$



$12\sqrt{2}.$

$$= \int_0^{2^{3/2}} \sqrt{2}\, x^{-1/3}\, dx \qquad \left[\sqrt{2}\,\frac{3}{2}\, x^{2/3}\right]_0^{2^{3/2}} = \sqrt{2}\,\frac{3}{2}\, 2 = 3\sqrt{2}$$

11.2 Q3.

$$c(t) = (\underset{x}{11t-16},\ \underset{y}{16t-1}).$$

find speed at $t=2$

units.

m/s.

$$c'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

$$c'(t) = (11, 16).$$

$$\text{speed} = \|c'(t)\| = \sqrt{11^2+16^2} = \sqrt{377} \approx 19.4$$

Q9

$$\sum_{n=2}^{\infty} \underbrace{\frac{\sqrt{n}}{n^2-1}}_{a_n}$$

large n $\sim$ $\frac{\sqrt{n}}{n^2} = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$.

$\sum \frac{1}{n^{3/2}} \leftarrow$ p-series w/ $p > 1$.

$\boxed{\text{Converges}}$

limit comparison.
(positive series only!).

$b_n = \frac{1}{n^{3/2}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2-1} \cdot n^{3/2} = \lim_{n\to\infty} \frac{n^2}{n^2-1}.$$

$$= \lim_{n\to\infty} \frac{1}{1-1/n^2} = 1. \qquad 0 < 1 < \infty$$

$\sum a_n$ converges $\Leftrightarrow$ $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converge.

Q10 $\quad f(x) = x\sin(2x).$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots + (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} + \cdots$$

$$x\sin(2x) = 2x^2 - \frac{2^3x^4}{3!} + \frac{2^5x^6}{5!} - \cdots + (-1)^n \frac{2^{2n+1}\cdot x^{2n+2}}{(2n+1)!} + \cdots$$

ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{\overset{2n+3.}{2(n+1)+1}}\cdot x^{\overset{2n+4}{2(n+1)+2}}}{(\underset{2n+3.}{2(n+1)+1})!}\cdot\frac{(2n+1)!}{2^{2n+1}\,x^{2n+2}}\right|.$$

$$\lim_{n\to\infty} \left| \frac{x^2 4}{(2n+2)(2n+3)} \right| . = 0 < \infty .$$

converges for all $|x|$, so $R = \infty$.

$(-\infty, \infty)$.

useful power series.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$\ln|1-x|$.

e^x, $\sin x$, $\cos x$.

$\frac{1}{1-x^2}$ . . .

$\tan^{-1}(x)$ . . .