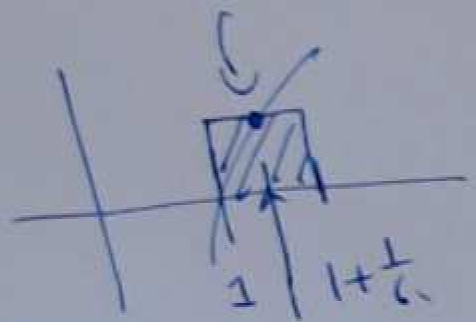
1

WW 5.1 Q ~~3~~ 2

$f(x) = \frac{7}{3}\ln(x^9)$ $[1,2]$

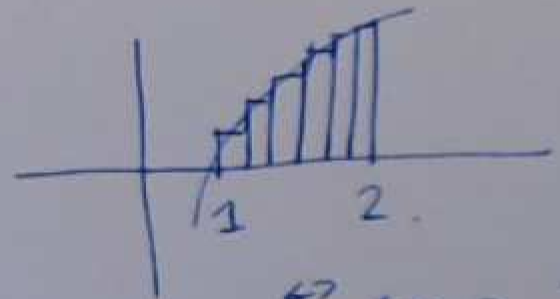
$= 21\ln(x)$

M_6. $f(\frac{13}{12})$



1 $1+\frac{1}{6}$

$1+\frac{1}{12}$



1 2.

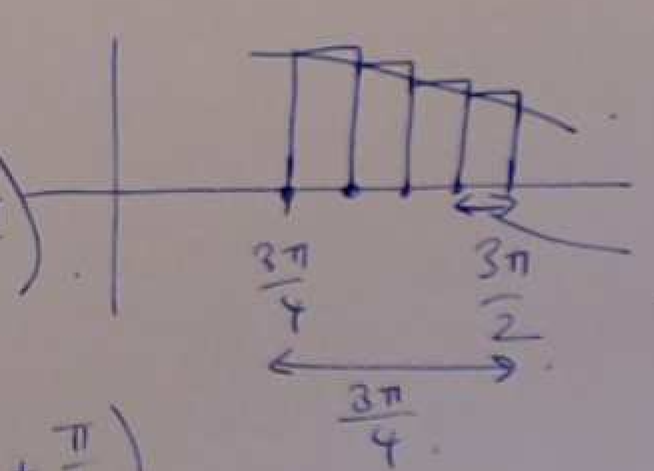
$\Delta x = \frac{1}{6}$.

$M_6 = \frac{1}{6} f(1+\frac{1}{12}) + \frac{1}{6} f(1+\frac{1}{6}+\frac{1}{12})$
$+ \frac{1}{6} f(1+\frac{2}{6}+\frac{1}{12}) + \cdots \frac{1}{6} f(1+\frac{5}{6}+\frac{1}{12})$

$M_6 = \frac{1}{6} \cdot 21 \Big(\ln(\frac{13}{12}) + \ln(\frac{15}{12}) + \ln(\frac{17}{12})$
$+ \ln(\frac{19}{12}) + \ln(\frac{21}{12}) + \ln(\frac{23}{12}) \Big)$

HW 5.1 Q3 L_4 $21\cos\left(\frac{x}{3}\right)$ $\left[\frac{3\pi}{4}, \frac{3\pi}{2}\right]$

$$L_4 = \frac{3\pi}{16} f\left(\frac{3\pi}{4}\right) + \frac{3\pi}{16} f\left(\frac{3\pi}{4}+\frac{3\pi}{16}\right) + \frac{3\pi}{16} f\left(\frac{3\pi}{4}+\frac{6\pi}{16}\right) + \frac{3\pi}{16} f\left(\frac{3\pi}{4}+\frac{9\pi}{16}\right)$$



$\frac{3\pi}{16} = \Delta x$

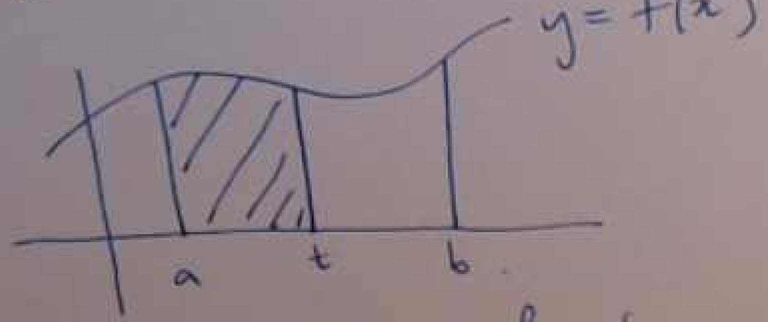
$$= \frac{3\pi}{16}\cdot 21\left(\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}+\frac{\pi}{16}\right) + \cos\left(\frac{\pi}{4}+\frac{2\pi}{16}\right) + \cos\left(\frac{\pi}{4}+\frac{3\pi}{16}\right)\right)$$

§5.4 Fundamental theorem of calculus

Thm (FTC ①) suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an antiderivative for $f(x)$.

Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

intuition



consider $\int_a^t f(x)\,dx$ ⊛

Q: what is the rate of change of ⊛ with respect to t?

$\frac{d}{dt}\int_a^t f(x)\,dx$.

$f'(\overset{t}{\cancel{x}}) = \lim_{h\to 0} \frac{f(\overset{t}{\cancel{x}}+h) - f(\overset{t}{\cancel{x}})}{h}$

$$\int_a^t f(x)\,dx \qquad \frac{d}{dt}\int_a^t f(x)\,dx = \lim_{h\to 0} \frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h}.$$

$$= \lim_{h\to 0} \frac{\int_t^{t+h} f(x)\,dx}{h}.$$

$$\approx \frac{\text{area of the rectangle}}{h} = \frac{h\,f(t)}{h} = f(t).$$

i.e. $\int_a^t f(x)\,dx$ is an antiderivative for $f(x)$,

so $\int_a^t f(x)\,dx = F(\cancel{x}\,t) + c$.   Q: what is the constant c?

and

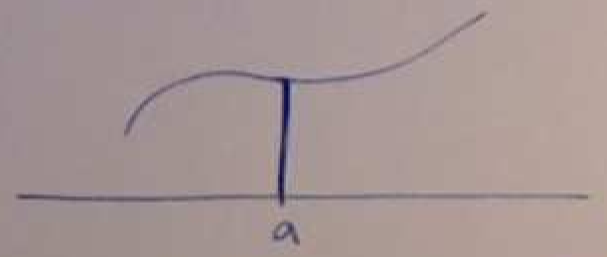
Then

intuition

Q what is the

$\frac{d}{dt}\int_a^t f(x)\,dx$.

$$\int_a^t f(x)\,dx = F(t) + c$$



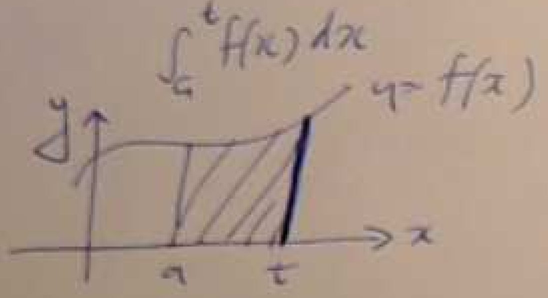
$t=a$:

$t=a$.

$$\int_a^a f(x)\,dx = 0$$

$$\underbrace{\int_a^a f(x)\,dx}_{=\,0} = F(a) + c$$

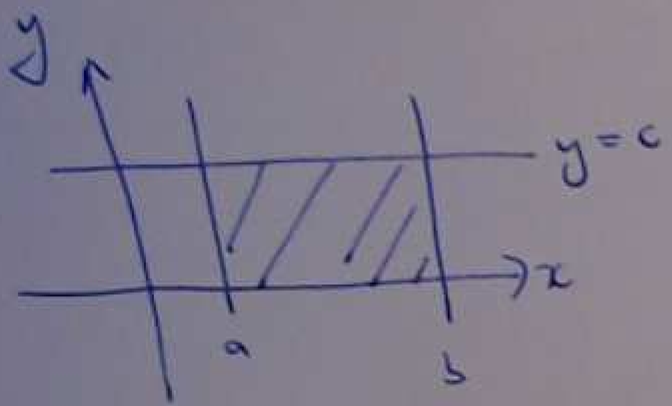
$$\Rightarrow \; -F(a) = +c.$$



so $$\int_a^t f(x)\,dx = F(t) - F(a)$$

$$\int_a^b f(x)\,dx = F(b) - F(a). \quad \square$$

Examples



ex $\int_a^b f(x)\,dx = F(b) - F(a)$

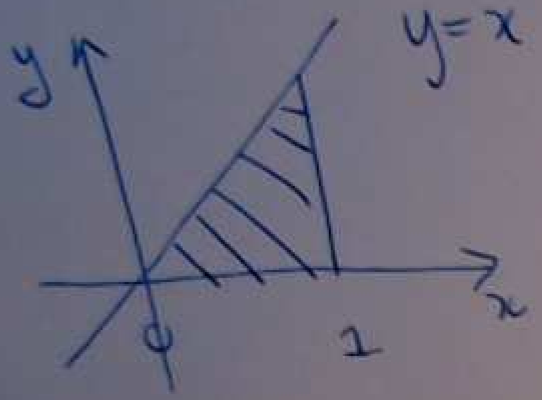
$$\int_a^b c \, dx$$

$$F(x) = cx$$

$$= F(b) - F(a)$$

$$= cb - ca$$

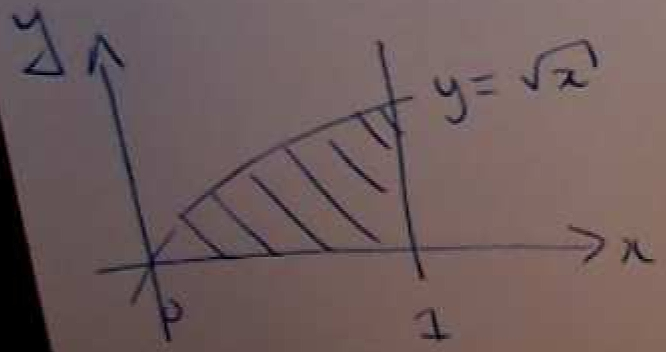
$$= c(b-a) \checkmark$$



$$\int_0^1 x\,dx = F(1) - F(0)$$

$$F(x) = \frac{1}{2}x^2$$

$$= \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$$

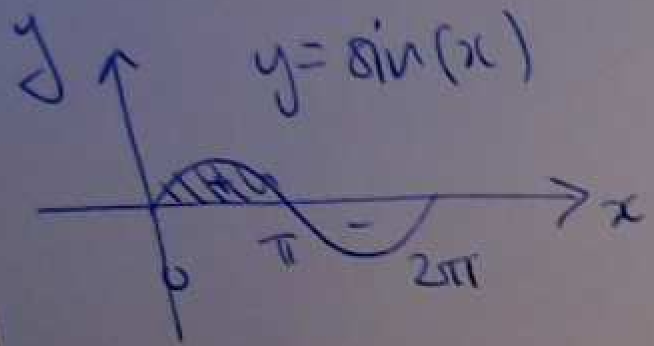


$$\int_0^1 \sqrt{x}\,dx = F(1) - F(0)$$

$$\sqrt{x} = x^{1/2}$$

$$F(x) = \frac{2}{3}x^{3/2}$$

$$= \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{2}{3}$$



$$\int_0^{2\pi} \sin(x)\,dx = 0.$$

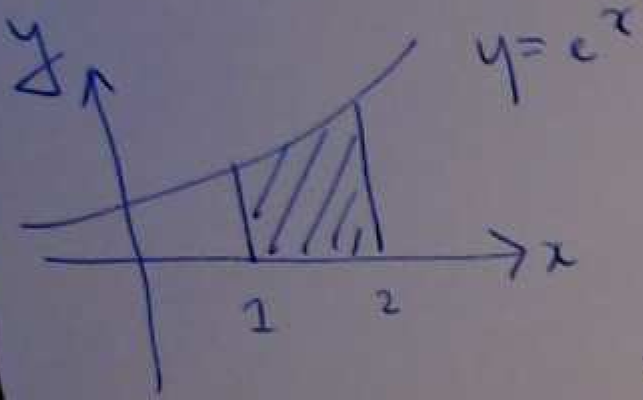
$$\int_0^{\pi} \sin(x) = F(\pi) - F(0).$$

$F(x) = -\cos(x)$

$$= -\cos(\pi) - -\cos(0).$$

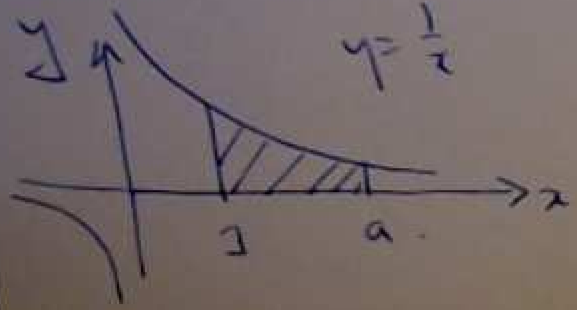
$$= -(-1) + \cos(0)$$

$$= 1 + 1 = 2$$



$$\int_1^2 e^x \, dx = F(2) - F(1)$$

$$F(x) = e^x \qquad = e^2 - e^1$$



$$\int_1^a \frac{1}{x} \, dx = F(a) - F(1)$$

$$F(x) = \ln|x|$$

$$= \ln(a) - \ln(1)$$

$$= \ln(a)$$

observations

1 choice of antiderivative doesn't matter.

let $F(x)$, $F(x)+c$ be antiderivatives for $f(x)$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= F(b)+c - (F(a)+c)$$

$$= F(b)+\cancel{c} - F(a) - \cancel{c}$$

$$= F(b) - F(a)$$

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(2)

$\int_a^t f(x)\,dx$ is a function of t.

x is a dummy variable.

$\parallel$

$\int_a^t f(y)\,dy$

$$\sum_{i=1}^{4} i = 1+2+3+4.$$

$$\sum_{j=1}^{4} j = 1+2+3+4.$$

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n.$$

if you want a function of x write

$\int_a^x f(t)\,dt$.

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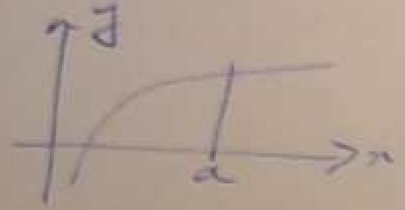
§5.5 Fundamental theorem of calculus II

Thm (FTC 2) Let $f(x)$ be a continuous function on $[a,b]$, then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative for $f(x)$, i.e. $A'(x) = f(x) = \frac{dA}{dx}$

i.e. $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Furthermore $A(a) = 0$.



□.

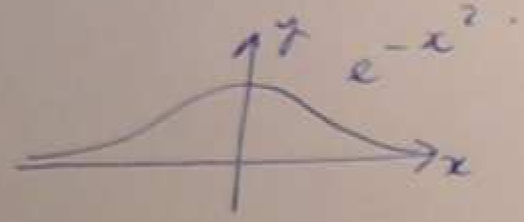
we have complete rules for differentiation.

formula $f(x)$; $\xrightarrow[\text{rules.}]{\text{apply}}$ $f'(x)$

elementary function

combination of x^n, x^r,

e^x, $\sin(x)$, $\cos(x)$, $\ln(x)$, ...



this doesn't work for integration

there are elementary functions. $\int_{-\infty}^{t} e^{-x^2}\,dx$

whose integral is <u>not</u> an elementary function.

recall solving equations.

① $x-4=0$.
$x = 4$ ✓.

② $x+1 = 0$
$\quad -1 \quad -1$
$x = -1$. ✓.

③ $x^2 - 2 = 0$.
$x^2 = 2$.
$x = \pm\sqrt{2} \neq \frac{p}{q}$

not a rational number!

④ $ax^2+bx+c = 0$.

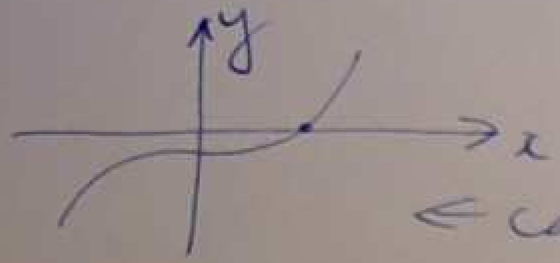
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

quadratic formula.

Fact there are (more complicated) formulae for deg 3 (cubic) deg 4 (quartic).

<u>Fact</u> there is no general formula for the roots of a deg 5 (quintic) polynomial.

$x^5 - x - 1$ ⊛



← this has a real solution

← can find a numerical approx.

there is no way of writing this number down in terms of $\sqrt[5]{\ }$ $a, b, \dots$ ← coeffs of ⊛.

[Galois theory]

analogues: $\int e^{-x^2}\,dx$, $\int \frac{\sin(x)}{x}\,dx$ ... $\int \sin(x^2)\,dx$

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§ 5.6 Substitution / change of variable

"reverse chain rule for integration".

recall chain rule $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x))\, g'(x)$.

substitution | change of variable

$\int f(x)\,dx\,, \quad x(u).$

$$\boxed{\int_{x=a}^{x=b} f(x)\,dx = \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\,\frac{dx}{du}\,du.}$$

$x(u)=a \quad u=x^{-1}(a)$

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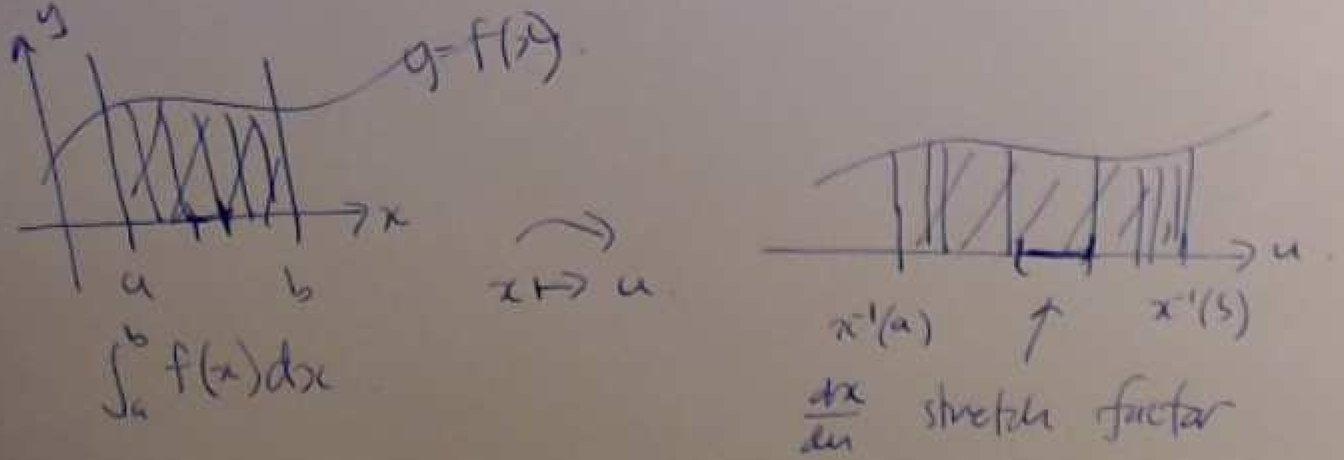
$$\int_a^b f(x)\,\underline{dx} = \int_{x^{-1}(a)}^{x^{-1}(b)} f(x(u))\,\underline{\frac{dx}{du}\,du}$$

<u>remember</u>: three things to change: function
differential
limits.

<u>mnemonic</u>: $\cancel{du}\ dx = \frac{dx}{du}\,du$ "cancelling fractions".



why does this work?

$$\int f(x)\,dx, \quad x(u) \qquad \text{set} \quad F(x) = \int f(x)\,dx$$

$$F'(x) = f(x)$$

$$\int f(x)\,dx = F(x) \underset{\text{wrt } u}{\overset{\text{diff}}{\rightsquigarrow}} \frac{d}{du} F(x(u)) = F'(x(u)) \cdot x'(u)$$

$$= \underset{\text{wrt } u}{\overset{\text{integrate}}{\rightsquigarrow}} \int F'(x(u))\, x'(u)\, du$$

$$= \int f(x(u)) \cancel{\frac{dy}{d}} \frac{dx}{du}\, du \qquad \square$$

useful fact: $\frac{dx}{du} = 1 \Big/ \frac{du}{dx}$.

Examples

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\int e^x \, dx = e^x + c$$

(1) $\displaystyle\int_{x=0}^{1} e^{-7x} \, dx$

set $u = -7x$. $\quad \frac{du}{dx} = -7$. $\quad \frac{dx}{du} = \frac{1}{-7}$.

$$\int_0^{-7} e^u \frac{dx}{du} \, du = \int_0^{-7} e^u \cdot -\frac{1}{7} \, du$$

$$= -\frac{1}{7} \int_0^{-7} e^u \, du = -\frac{1}{7}\left[e^{-7} - e^0\right] = \frac{1}{7}\left(1 - e^{-7}\right)$$

2 $\displaystyle\int_{\underbrace{0}_{3}}^{2} \underbrace{x^2\sqrt{x^3+1}}_{1}\ \underbrace{dx}_{2}$

try

$u = x^3 + 1$

$\dfrac{du}{dx} = 2x^2$

$\dfrac{dx}{du} = \dfrac{1}{du/dx}$

$$\int_1^9 x^2\sqrt{u}\ \frac{dx}{du}\ du$$

$$\int_1^9 x^2\sqrt{u}\ \frac{1}{2\cancel{x^2}}\ du = \int_1^9 \frac{1}{2} u^{1/2}\ du$$

$$= \left[\frac{1}{2}\frac{2}{3}u^{3/2}\right]_1^9 = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{1}{3}(27-1) = \frac{26}{3}$$

$$\cancel{\frac{d}{dx}\left(\sqrt{x^3+1}\right) = \frac{d}{dx}\left(\left(x^3+1\right)^{1/2}\right)}$$

$$\cancel{= \frac{1}{2}\left(x^3\right.}$$

$$\frac{d}{dx}\left(\left(x^3+1\right)^{3/2}\right) = \frac{3}{2}\left(x^3+1\right)^{1/2} \cdot 3x^2$$

$$= \frac{9}{2}x^2\sqrt{x^3+1}$$

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(2) $$\underbrace{\int}_{③}{}_0^2 \underbrace{x^2\sqrt{x^3+1}}_{①} \underbrace{dx}_{②}$$

try

$$u = x^3 + 1$$

$$\frac{du}{dx} = 3x^2$$

$$\int_1^9 x^2\sqrt{u}\,\frac{dx}{du}\,du \qquad \frac{dx}{du} = \frac{1}{du/dx}$$

$$\int_1^9 x^2\sqrt{u}\,\frac{1}{3\cancel{x^2}}\,du = \int_1^9 \frac{1}{3}u^{1/2}\,du$$

$$= \left[\frac{1}{3}\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{9}\left(9^{3/2} - 1^{3/2}\right) = \frac{2}{9}(27-1) = \frac{26\times 2}{9}$$

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