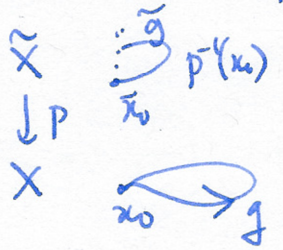
Prop$^{\underline{n}}$ Let $p:\tilde{X}\to X$ be a covering space, with $X, \tilde{X}$ path connected. Then $\deg(p) =$ index of $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ in $\pi_1(X,x_0)$ $[p_*(\pi_1(\tilde{X},\tilde{x}_0)), \pi_1(X,x_0)]$.

Proof



$g: I\to X$ loop based at x_0

$\tilde{g}: I\to\tilde{X}$ path based at $\tilde{x}_0$ ending at $g(1)\in p^{-1}(x_0)$

set $H = p_*(\pi_1(\tilde{X},\tilde{x}_0))$ note: if $[h]\in H$, then $\tilde{h}$ is a loop so $\tilde{h}: I\to\tilde{X}$ ends at $\tilde{x}_0\in p^{-1}(x_0)$.

so for any $[h]\in H$, $h\cdot g$ has a lift $\tilde{h}\cdot\tilde{g}$ ending at $g(1)$, as $\tilde{h}$ is a loop based at $\tilde{x}_0$, so we can define a function Φ: cosets of $H \longrightarrow p^{-1}(x_0)$

$$H[g]\longmapsto \tilde{g}(1)$$

claim: Φ is surjective: $\tilde{X}$ path connected, so there is a path connecting $\tilde{x}_0$ to any other point in $p^{-1}(x_0)$

Φ is injective: suppose $\Phi(H[g_1]) = \Phi(H[g_2])$, then $\tilde{g}_1(1) = \tilde{g}_2(1)\in p^{-1}(x_0)$

so $g_1\bar{g}_2$ is a loop in $\tilde{X}$ based at x_0, so $[g_1][g_2]^{-1}\in H$, so

$H[g_1] = H[g_2]$ $\square$.

Q: when do maps lift? A lifting criterion/property

Prop$^{\underline{n}}$ let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering space, and let $f:(Y,y_0)\to(X,x_0)$, Y path connected, locally path connected. Then a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ exists iff $f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

$f_*(\pi_1 Y)\subseteq p_*(\pi_1\tilde{X})$.

Def$^{\underline{n}}$ X is locally path connected if for all $x\in X$, and each open nbd U s.t. $x\in U\subseteq X$, there is an open nbd V, $x\in V\subseteq U$ w/ V path connected.

Proof $\Rightarrow$ if the lift exists, then $p\tilde{f}=f$, so $p_*\tilde{f}_* = f_*$, so $f_*(\pi_1 Y) = p_*\tilde{f}_*(\pi_1 Y)\subset p_*(\pi_1\tilde{X})$.