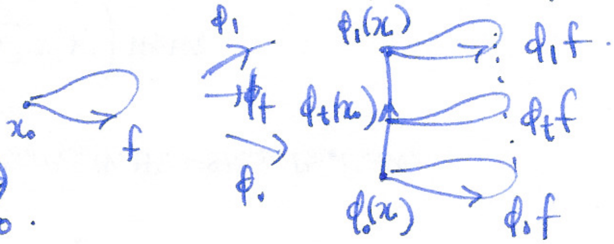
Lemma If $\phi_t : X \to Y$ is a homotopy and h is the path $h(t) = \phi_t(x_0)$

then
$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(\phi_1)_*"] \arrow[dr, "(\phi_0)_*"'] & \pi_1(Y, \phi_1(x_0)) \arrow[d, "\beta_h"] \\ & \pi_1(Y, \phi_0(x_0)) \end{tikzcd}$$
commutes, i.e. $\beta_h(\phi_1)_* = (\phi_0)_*$, where β_h is the change of basepoint map.

Proof Let $f : I \to X$ be a loop in X



want: $[\phi_0 f] = \beta_h[\cancel{\beta_h}\, \phi_1 f]$, where $h = \phi_t|_{x_0}$.

Let $h_t(s) = h(ts)$

$h_t : \underset{\text{stretch}}{[0,t] \longrightarrow [0,1]} \longrightarrow Y$. Then $h_t \cdot \phi_t f \cdot \overline{h}_t$ is a homotopy of loops

based at $\phi_0(x_0)$. So $h_0 \cdot f \cdot \overline{h}_0 \simeq h_1 \cdot \phi_1 f \cdot \overline{h}_1 = h \cdot \phi_1 f \cdot \overline{h}$.

so $(\phi_0)_*[f] = \beta_h((\phi_1)_*[f])$. $\square$.

Proof (of Prop^{n}) $X \underset{\psi}{\overset{\phi}{\rightleftarrows}} Y$ homotopy equivalence

consider:
$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \phi\psi\phi(x_0))$$

$\psi\phi \simeq id_X \Rightarrow \psi_*\phi_* = \beta_h \mathbb{1}_{\pi_1 X}$, isomorphism $\Rightarrow$ ϕ_* injective, ψ_* surjective.

similarly $\phi\psi \simeq id_Y \Rightarrow \phi_*\psi_*$ is an iso, so ψ_* inj. ϕ_* surj. so ϕ_* (and ψ_*) are isomorphism $\square$.

§1.2 Van Kampen's Th^{m}

Free products.

example $F_2 = \langle a, b \mid \rangle$ set W = all finite strings of $\{a, a^{-1}, b, b^{-1}\}$.

e.g. $\phi \in W$ $a \in W$, $aba b^{-1} \in W$ $aabb^{-1}abb \in W$ etc.

define an equivalence relation $\sim$ on W by
$$u a a^{-1} v \sim uv \qquad \text{for all } u, v \in W$$
$$u a^{-1} a v \sim uv$$
$$u b b^{-1} v \sim uv$$
$$u b^{-1} b v \sim uv$$

notation: $\underline{aa\ldots a} = a^n$, etc.

$W/\sim$ may be identified with set of <u>reduced words</u> (minimal length elements of equivalence classes).

<u>multiplication</u>: given by concatenation, i.e. $u \cdot v = uv$

e.g. $(aba)(a^{-1}b^{-1}aa) = ab\cancel{a}\cancel{a^{-1}}\cancel{b}\cancel{b^{-1}}aa \approx aaa = a^3$

<u>Exercise</u> this defines a group.

This is called the free group on 2 generators F_2. (identity element $\phi = 1$).

<u>Generalizations</u> • F_n free group on n-generators, i.e. W = all finite strings of $\{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$.

• Free products: let G, H be groups, the free product ~~...~~ $G * H$ consists of all strings / words of the form $a_1 a_2 \dots a_n$, $a_i \in G$ or H, up to the following equivalences: • if a_i, a_{i+1} both in the same group then $u a_i a_{i+1} v \sim uv$

• 1 in either group corresponds to ϕ.

a minimal length representative of an equivalence class is called a <u>reduced word</u>

$G * H$ is a group; • multiplication is concatenation

$$(a_1 \dots a_n)(b_1 \dots b_m) = a_1 \dots a_n b_1 \dots b_m \leftarrow \text{may not be reduced!}$$

• identity is ϕ.

• inverses: $(a_1 a_2 \dots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \dots a_1^{-1}$.

<u>Examples</u> • $\mathbb{Z} * \mathbb{Z} = F_2$, • $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ ← free product not free gp. $PSL_2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

<u>Universal property</u> $\ast_\alpha G_\alpha$ (arbitrary) free product

any collection of homomorphisms $\phi_\alpha : G_\alpha \to H$ extends to a unique homomorphism $\phi: \ast_\alpha G_\alpha \to H$.

<u>Group presentations</u> $G = \langle \underbrace{a_1, \dots, a_n}_{\text{generators}} \mid \underbrace{r_1, r_2, \dots, r_m}_{\text{relations}} \rangle$

$G = F_n / N$ F_n = free group generated by $a_1, \dots, a_n$

N = normal subgp generated by $r_1, \dots, r_m$.

Examples $F_2 = \langle a,b \mid \rangle$ $\quad \mathbb{Z} = \langle a \mid \rangle$ $\quad \mathbb{Z}^2 \cong \mathbb{Z}\oplus\mathbb{Z} = \langle a,b \mid aba^{-1}b^{-1}\rangle$ $\quad \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a,b \mid a^2, b^2\rangle$.

Warning ~~Given a group presentation~~ In the collection of fin. pres. grps, there is no algorithm:

- to decide if the group is trivial.
- to decide if a given word is trivial.

Abelianization For any group G, there is an abelian group $ab(G)$ s.t. any homomorphism $\phi: G \to A$ (abelian) factors through $ab(G)$

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & A \\ & ab(G) \arrow[ur] & \end{tikzcd}$$

Given a presentation of G, you can find $ab(G)$ by abelianizing the presentation.

e.g. $F_2 = \langle a,b \mid \rangle$ $\quad ab(F_2) = \langle a,b \mid ab = ba\rangle \cong \mathbb{Z}^2$

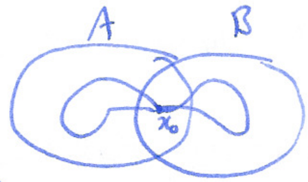
useful fact $ab(G) \not\cong ab(H) \Rightarrow G \not\cong H$

Corollary $F_n \not\cong F_m$ for $n \neq m$.

Warning . not nec. easy to find presentation. $PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

Q: $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \;\middle|\; -?\right\rangle$. $\quad$ Q: what is $\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right\rangle$?

van Kampen idea: $X = A \cup B$



A, B path connected

$A \cap B$ path connected

$A \hookrightarrow A \cup B \qquad B \hookrightarrow A \cup B$

$\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \qquad \pi_1(B, x_0) \xrightarrow{i_*} \pi_1(X, x_0)$

$\pi_1(A, x_0) * \pi_1(B, x_0) \underset{\Phi}{\twoheadrightarrow} \pi_1(X, x_0)$.

- Φ surjective!
- $\ker(\Phi)$ = normal subgp generated by $i_\alpha(\omega)\, i_\beta(\omega)^{-1}$.

$$\begin{tikzcd} & A \arrow[dr, hook, "i_\alpha"] & \\ A \cap B \arrow[ur, hook] \arrow[dr, hook] & & A \cup B = X \\ & B \arrow[ur, hook, "i_\beta"'] & \end{tikzcd}$$

Thm (two set version) $X = A \cup B$, basepoint $x_0 \in A \cap B$, A, B open, path connected, $A \cap B$ path connected, then $\Phi: \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$ is surjective, with kernel normal subgp N generated by all elements of the form $i_\alpha(\omega)\, i_\beta(\omega)^{-1}$, i.e. $\pi_1(X, x_0) \cong \pi_1(A, x_0) * \pi_1(B, x_0) / N$.