Slopes of exponential functions

consider $f(x) = b^x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h\to 0} b^x \left(\frac{b^h - 1}{h}\right) = b^x \underbrace{\lim_{h\to 0} \frac{b^h-1}{h}}_{\text{assume this limit exists, call it } m_b}$$

So for exponential functions: slope is proportional to value of function

$f(x) = b^x$ then $f'(x) = m_b \cdot b^x$

<u>note</u> slope at 0 is $f'(0) = m_b b^0 = m_b$

<u>recall</u> e is the number such that slope of e^x at $x=0$ is equal to 1. So $f(x) = e^x$ then $f'(x) = e^x$

$$\frac{d}{dx}(e^x) = e^x$$

<u>Example</u> $f(x) = 7e^x + 4x^2$

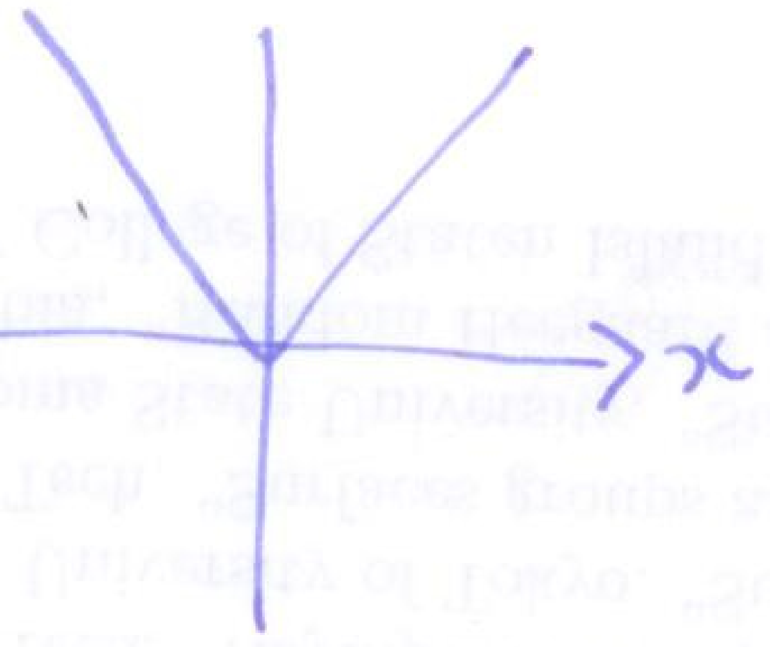
$f'(x) = 7e^x + 8x$

<u>Useful fact</u>: differentiable $\Rightarrow$ continuous.

warning: continuous $\not\Rightarrow$ differentiable

Example continuous but not differentiable

$f(x) = |x|$

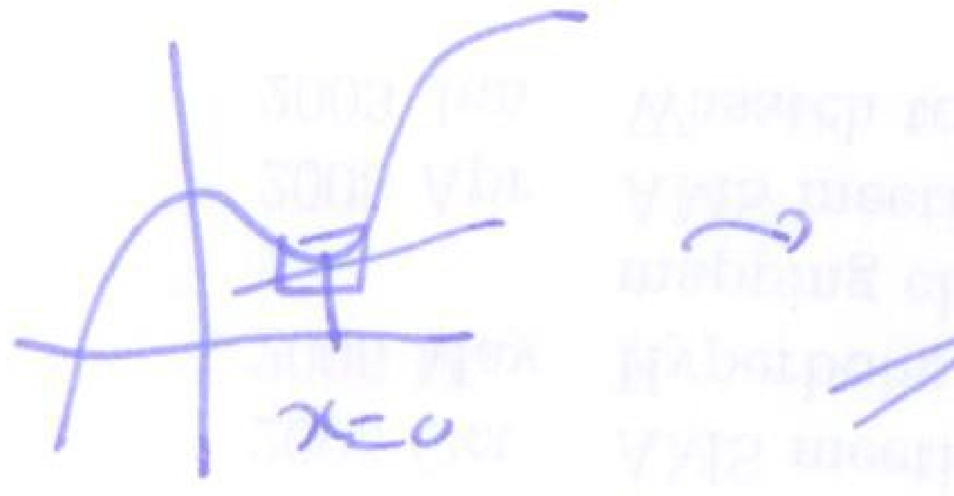


continuous ✓

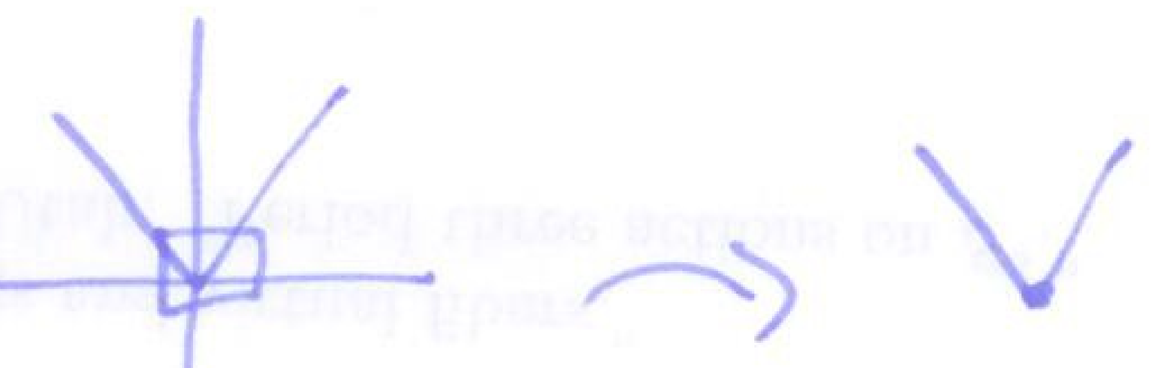
differentiable ? $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{|0+h| - |0|}{h} = \lim\limits_{h \to 0} \dfrac{|h|}{h}$

$$\left.\begin{aligned} \lim_{h \to 0+} \frac{|h|}{h} &= \lim_{h \to 0+} \frac{h}{h} = \lim_{h \to 0} 1 = 1 \\ \lim_{h \to 0-} \frac{|h|}{h} &= \lim_{h \to 0-} \frac{-h}{h} = \lim_{h \to 0} -1 = -1 \end{aligned}\right\} \text{not equal} \quad \text{so } \lim_{h \to 0} \frac{|h|}{h} \text{ DNE.}$$

__local picture__ : if $f(x)$ is differentiable at $x = c$, then if you look closely enough, it looks close to a straight line



differentiable



not differentiable.

§3.3 Product and quotient rules

new functions from old: $f(x)g(x)$ (product) $\qquad f(x)/g(x)$ (quotient)

Thm: Product rule Suppose f, g are differentiable at x

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

Warning $(fg)' \neq f'g'$!!

Example

① $$\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x)\cdot x + x \cdot \frac{d}{dx}(x)$$
$$= 1x + x \cdot 1$$
$$= 2x.$$

② $$\frac{d}{dx}\left(3x^2(x^2+1)\right) = \frac{d}{dx}(3x^2)\cdot(x^2+1) + 3x^2 \cdot \frac{d}{dx}(x^2+1)$$
$$= 6x(x^2+1) + 3x^2 \cdot 2x$$
$$= 6x^3 + 6x + 6x^3 = 12x^3 + 6x$$

③ $$\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x)$$
$$= 2xe^x + x^2e^x = (2x+x^2)e^x$$

Proof (of product rule)

assume f, g both differentiable at x.

$$(fg)'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

trick: $$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} \cdot g(x)$$

sum: $$\lim_{h \to 0} f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} g(x)$$

products: $$\lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x)$$

$$= f(x) g'(x) + f'(x) g(x)$$

Thm Quotient rule

suppose f, g are differentiable at x, then $f(x)/g(x)$ is differentiable for all x s.t. $g(x) \neq 0$. and

$$\left(\frac{f}{g}\right)' = \frac{g(x) f'(x) - f(x) g'(x)}{g(x)^2} \qquad \square$$

Example ① $$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{x \cdot (1)' - (1)(x)'}{x^2} = \frac{0 - 1}{x^2} = -\frac{1}{x^2}$$

Example ② $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - (x+1)'(x)}{(x+1)^2}$

$$= \frac{(x+1)\cdot 1 - 1\cdot x}{(x+1)^2} = \frac{x+1-x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

③ $\frac{d}{dt}\left(\frac{e^t}{e^t+1}\right) = \frac{(e^t+1)(e^t)' - (e^t+1)'(e^t)}{(e^t+1)^2}$

$$= \frac{(e^t+1)e^t - e^t\cdot e^t}{(e^t+1)^2} = \frac{e^{2t}+e^t-e^{2t}}{(e^t+1)^2} = \frac{e^t}{(e^t+1)^2}$$

Example

① a ball thrown in the air moves as $h(t) = 20t - 5t^2$

what is its maximum hight? $\longleftrightarrow$ slope $= 0$.

$h'(t) = 20 - 10t$

solve $20 - 10t = 0$

$20 = 10t$ $\quad t = 2$

height: $h(2) = 20.2 - 5.(2)^2 = 40 - 20 = 20$ m.