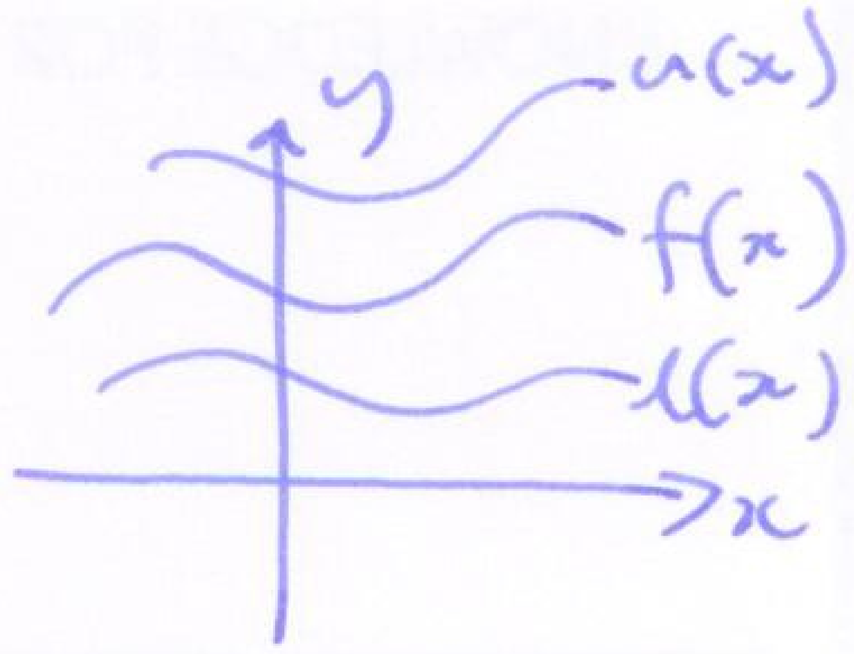
§2.6 Trigonometric limits

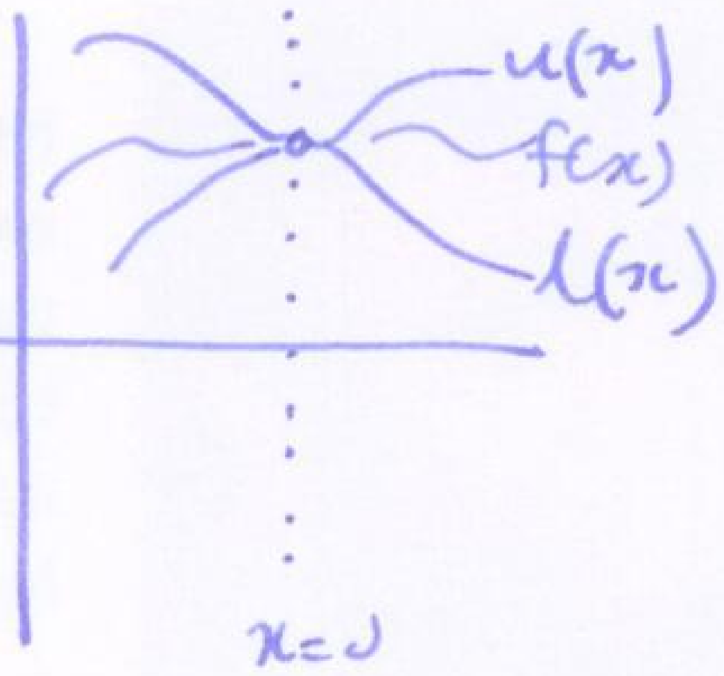
Q: what is $\lim_{x \to 0} \frac{\sin x}{x}$ note $\frac{\sin(0.01)}{0.01} \sim 1$

Squeeze theorem suppose $l(x) \le f(x) \le u(x)$



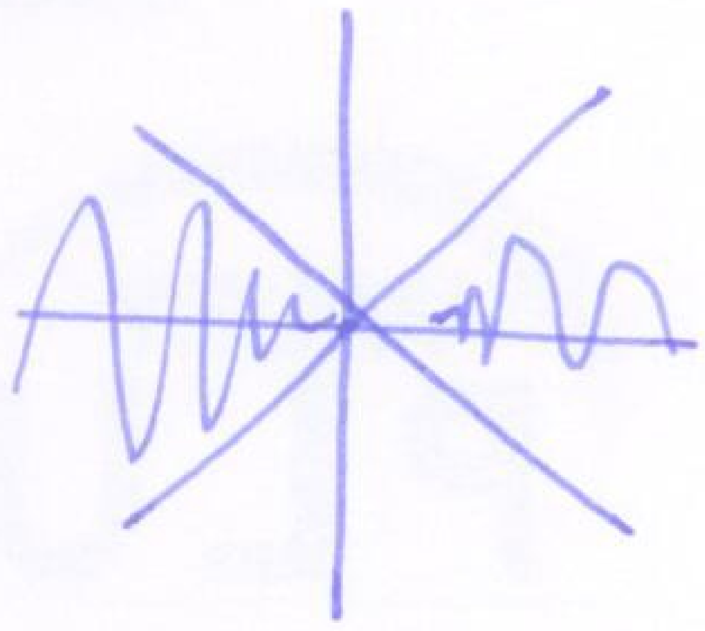
and $\lim_{x \to c} l(x) = L = \lim_{x \to c} u(x)$

then $\lim_{x \to c} f(x) = L$



Example $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$

note: $-1 \le \sin(\theta) \le 1$ so $-|x| \le x \sin(1/x) \le |x|$



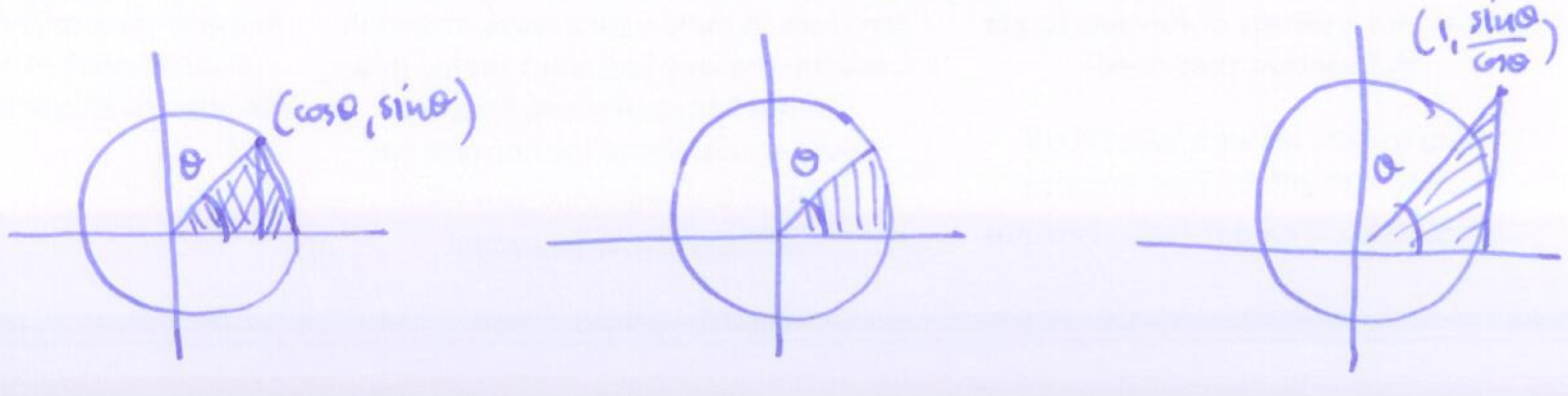
$\lim_{x \to 0} |x| = 0$ $\lim_{x \to 0} -|x| = 0$

$\Rightarrow \lim_{x \to 0} x \sin(1/x) = 0$

Thm $\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Proof $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ assume $0 < \theta < \frac{\pi}{2}$, consider the following

three areas:



area of triangle $= \frac{1}{2}bh$ $\leq$ area of sector $\leq$ area of triangle $\frac{1}{2}bh$

$= \frac{1}{2} 1 \sin\theta$ $\qquad \frac{\theta/2\pi}{\pi} = \frac{1}{2}\theta$ $\qquad \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

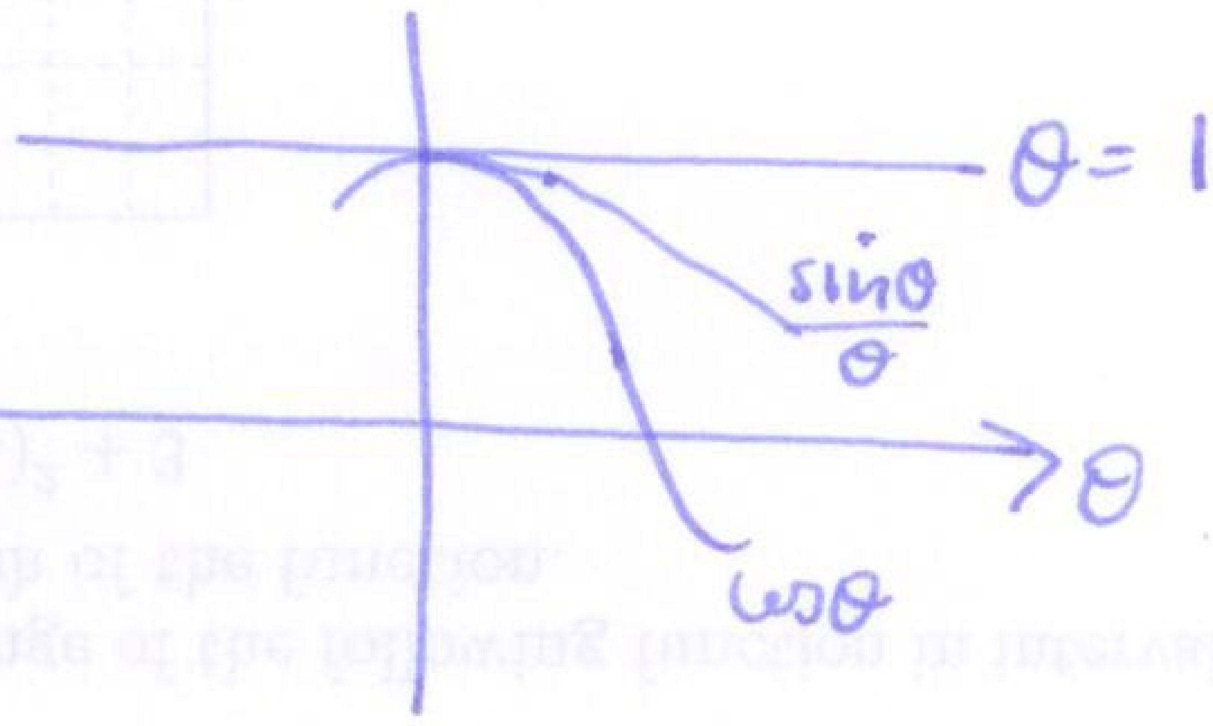
$$\frac{\sin\theta}{\theta} \leq 1$$

$$\theta \leq \frac{\sin\theta}{\cos\theta}$$

$$\cos\theta \leq \frac{\sin\theta}{\theta}$$

so

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$



$$\left.\begin{array}{l} \lim\limits_{\theta\to 0} \cos\theta = 1 \\ \lim\limits_{\theta\to 0} 1 = 1 \end{array}\right\} \Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Example

① $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{2x}$ $\qquad$ know: $\displaystyle \lim_{x\to 0}\frac{\sin\theta}{\theta} = 1$

write this as $3x = \theta \iff x = \theta/3$.

$$\lim_{\theta\to 0}\frac{\sin\theta}{2\theta/3} = \frac{3}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{3}{2}$$

② $$\lim_{t\to 0}\frac{1-\cos t}{\sin t} = \lim_{t\to 0}\frac{1-\cos t}{t}\cdot\frac{t}{\sin t}$$

note $\displaystyle \lim_{t\to 0}\frac{1-\cos t}{t} = 0$ $\qquad \displaystyle \lim_{t\to 0}\frac{\sin t}{t} = 1$ so ⊛ $= 0.1 = 0$.