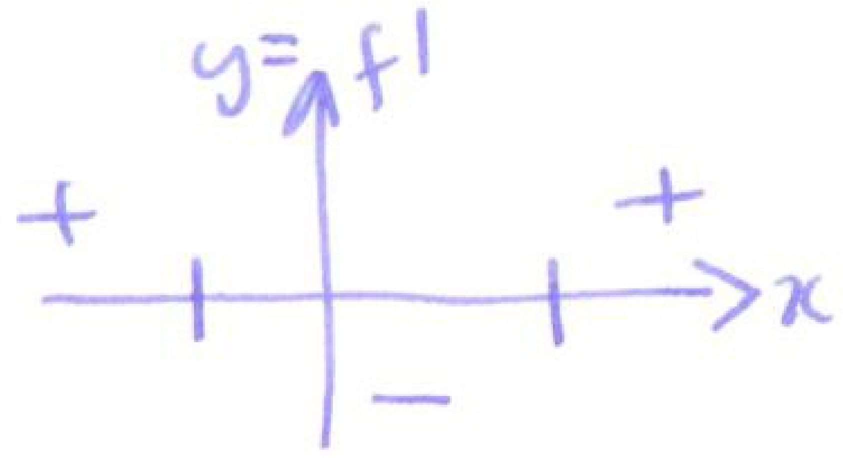
§4.5 Graph sketching and asymptotes

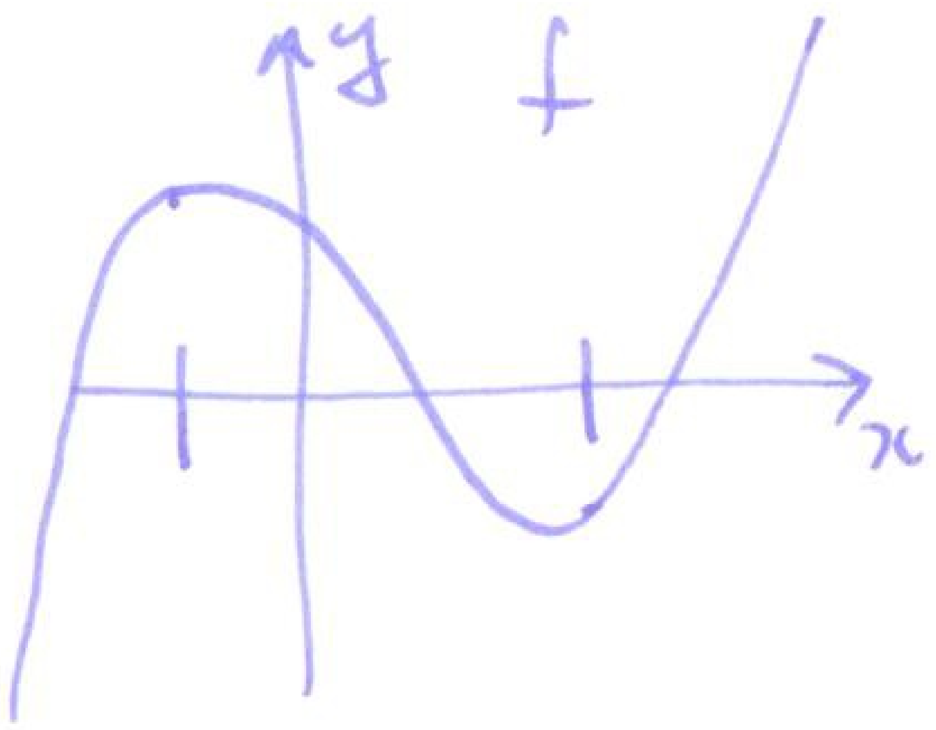
Example sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$. (i.e. find critical points and sign of f', f'').

$f'(x) = x^2 - x - 2 = (x+1)(x-2)$ critical points at $x = -1, 2$



$f''(x) = 2x - 1$ $\quad f''(-1) = -3 < 0$ max

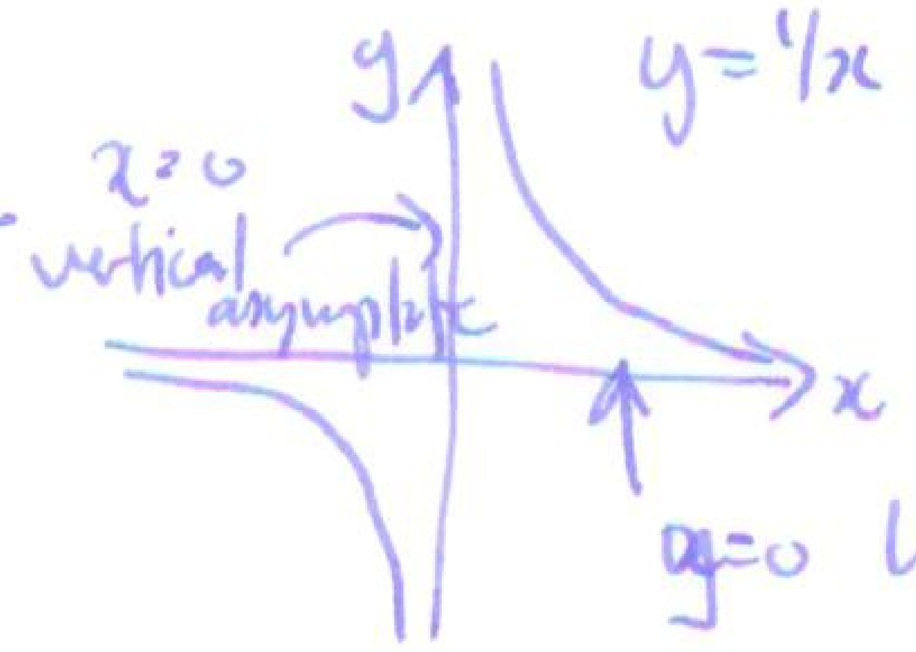
$f''(2) = 3 > 0$ min.



$f(-1) = -\frac{1}{3} - \frac{1}{2} + 2 + 3 > 0$

$f(2) = \frac{8}{3} - 2 - 4 + 3. < 0$

Asymptotes



$y = c$ is a horizontal asymptote if $\lim_{x\to\pm\infty} f(x) = c$ or $\lim_{x\to\infty} f(x) = c$

$x = c$ is a vertical asymptote if $\lim_{x\to c^\pm} f(x) = +\infty$ or $-\infty$.

~~Example~~ observation: rational functions $\frac{p(x)}{q(x)}$ can have horizontal asymptotes ~~limits~~ iff $\deg(p) \leq \deg(q)$

Example $\dfrac{x^2+x+1}{3x^2+2} = \dfrac{1 + 1/x + 1/x^2}{3 + 2/x^2} \longrightarrow \dfrac{1}{3}$ as $x \to \pm\infty$.

$\dfrac{x^3+x}{x^2-2} = \dfrac{x + 1/x}{1 - 2/x^2} \longrightarrow$ $+\infty$ as $x \to \infty$, $-\infty$ as $x \to -\infty$.

Example Identify critical points of $f(x) = (4x - x^2)e^x = 4xe^x - x^2e^x$.

find critical points: $f'(x) = 4xe^x + 4e^x - x^2e^x - 2xe^x$.

$$= (4x + 4 - x^2 - 2x)e^x$$

$$= (4 + 2x - x^2)e^x$$

solve $f'(x) = 0$ $\quad x^2 - 2x - 4 = 0 \quad x = \frac{2 \pm \sqrt{4+16}}{2} = 1 \pm \sqrt{5}$

now find $f''(x)$. $= (4 + 2x - x^2)e^x + (2 - 2x)e^x$

$$= (6 - x^2)e^x$$

$f''(1+\sqrt{5}) < 0$ (local max)

$f''(1-\sqrt{5}) > 0$ (local min)

Example (failure) $f(x) = x^5 - 5x^4$.

$f'(x) = 5x^4 - 20x^3$ solve $f'(x) = 0$ $\quad \cancel{5(x}$

$$5x^3(x-4) = 0$$

$f''(x) = 20x^3 - 60x^2 = \cancel{4}\, 20x^2(x-3)$ $\quad x = 0, 4$.

$f''(0) = 0$ $\quad f''(4) > 0$ (local min)

no info.
(use first derivative test)

Example sketch graph of $\frac{3x+2}{2x-4}$.

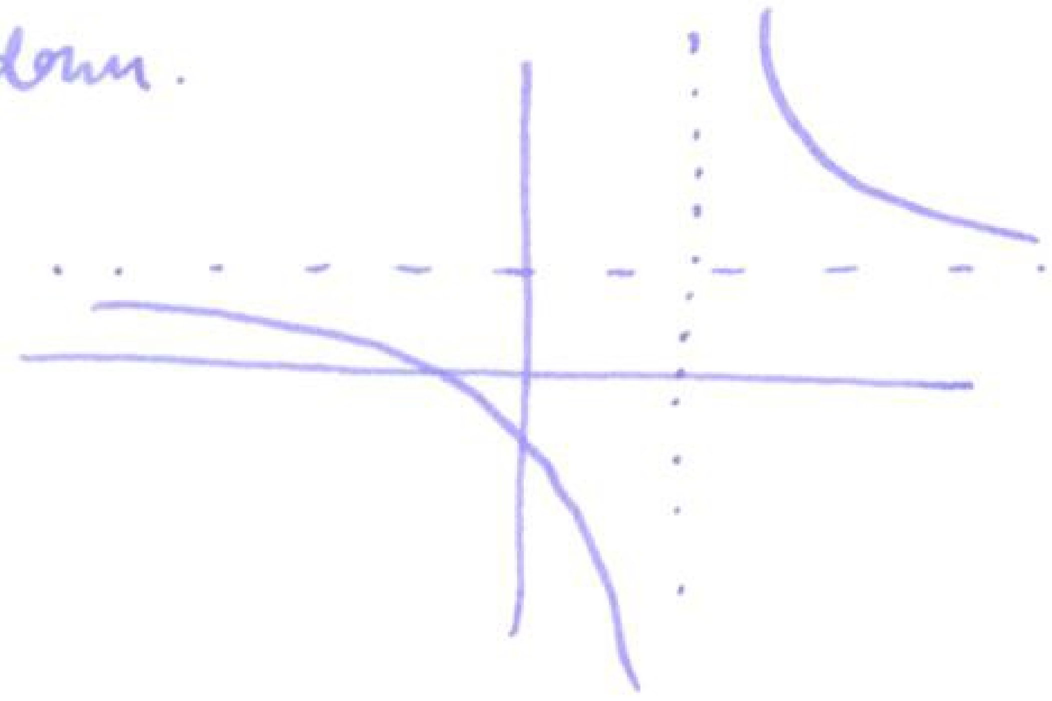
① find vertical asymptotes (i.e. when denominator is zero.)

$2x-4=0 \quad x=2$.

② find $f'(x) = \frac{-4}{(x-2)^2}$ $f''(x) = \frac{8}{(x-2)^3}$ no critical points.

−ve: decreasing   +ve. concave down.

③ horizontal asymptotes at $3/2$.



Example sketch graph of $\frac{x}{\sqrt{x^2+1}}$ $\cancel{\frac{x^2}{\sqrt{x^4-2x^2+1}}}$. $x(x^2+1)^{-1/2}$.

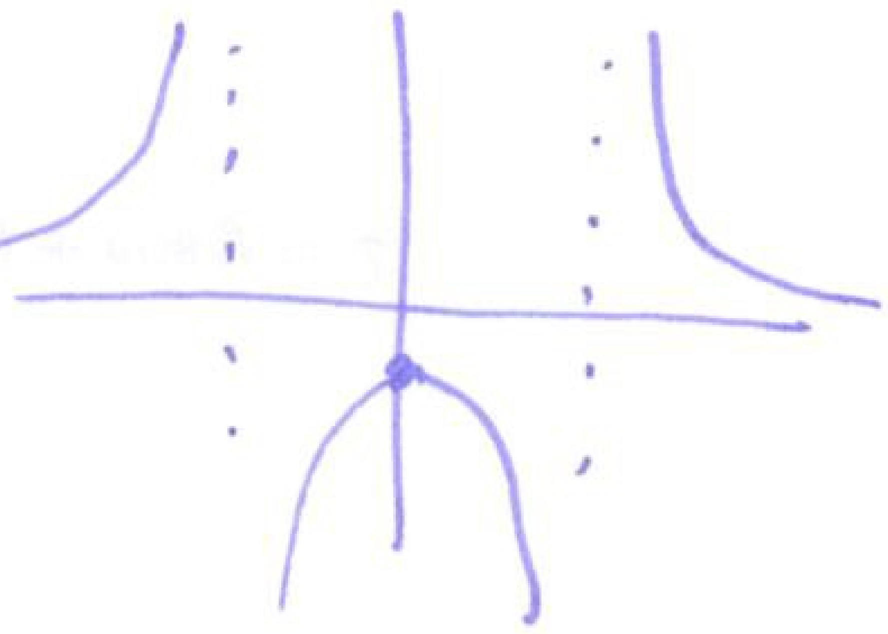
① vertical asymptotes: none.

② horizontal asymptotes: $\frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+1/x^2}} \longrightarrow 1$ as $x \to +\infty$.
$\longrightarrow -1$ as $x \to -\infty$

③ $f'(x) = \frac{\sqrt{x^2+1} - x\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{x^2+1} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0$

increasing

Example $\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$



$f'(x) = -\frac{2x}{(x^2-1)^2}$ $f''(x) = \frac{6x^2+2}{(x^2-1)^3}$